5.3 Series Solutions Near an Ordinary Point, Part II

- Recall the second order ODE;

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials. Then, we consider two questions:

1. if x_0 is an ordinary point of (1), there are power series solutions

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

which converges for $|x - x_0| < \rho$.

2. the radius of the convergence of such a series.

Theorem

If x_0 is an ordinary point of the DE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of the DE is

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0.
Further, the minimum of the radii of convergence for p and q enables us to determine a lower bound for the radius of convergence of series solutions y_1 and y_2.

- **How to determine the radius of convergence of power series for Q/P (or R/P):**

1. Consider the rational function Q/P which does not have common factors except one.
2. Find the zeros of the bottom P. The zeros may be complex numbers.
3. The radius of convergence of the power series about the point x_0 is the distance from x_0 to the nearest zero of P.

- Note that a series solution may converge in a wider range of x than indicated by the previous Theorem.

Examples

1. What is the radius of convergence of the Taylor series for $\left(1+x^2\right)^{-1}$ about $x_0 = 0$.

2. What is the radius of convergence of the Taylor series for $\left(x^2-2x-3\right)^{-1}$ about $x_0 = 2$?

Examples

Example

1. Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$\left(1 - x^2\right) y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

2. Determine a lower bound for the radius of convergence of series solutions about $x = 0$ or $x = 2$ for the DE

$$\left(1 + x^3\right) y'' + (1 + x)y' + y = 0.$$

3. Can we determine a series solution about $x = 0$ for the following DE?:

$$y'' + y' \cos x + \left(1 + x^2\right) y = 0.$$

If so, what is the radius of convergence?