7 Series Solutions of Linear Second Order Equations

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ASU

1. **Review of Power Series**
2. **Series Solution Near an Ordinary Point, Part I**

- In this Chapter, we study much larger class of equations which has variable coefficients (coefficient functions). In order to solve those DEs, we require the representation of a given function by a **power series**.
- We seek series representation for solutions:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{1}$$

If $P_0(x) \neq 0$, then solutions of (1) can be written as power series

$$y(x) = \sum_{n=0} a_n (x - x_0)^n ,$$

where it converges in an open interval of the center $x = x_0$.

- A power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to converge at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n (x - x_0)^n \quad \text{exits.}$$

It may converge for all x or it may converge for some values of x and not for others.

Theorem

(pp. 307) For the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ exactly one of the following statements is true:
(1) The power series converges only for $x = x_0$.
(2) The power series converges for all value of x, on $(-\infty, \infty)$.
(3) There is a positive number R called the radius of convergence such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.
Note that (2) gives $R = \infty$.

Ratio Test

- If for a **fixed value** of x

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}\left(x - x_0\right)^{n+1}}{a_n\left(x - x_0\right)^{n}} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad (2)$$

then the series converges absolutely if $R < 1$. The series diverges if $R > 1$ and the test fails (is inconclusive) if $R = 1$.

Theorem

(7.1.4) A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with $R > 0$ has derivatives of all differential orders in its open interval of convergence and successive derivatives can be obtained by repeatedly differentiating term by term:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

$$\dots$$

Shifting the summation index in a power series

- For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n (x - x_0)^{n-k} \tag{3}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k} (x - x_0)^n ;$$

that is , the power $n - k$ in (3) is shifted **up** by k and the initial index is shifted **down** by k.

Theorem

(7.1.6)

(a) If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

for all $x \in (x_0 - \rho, x_0 + \rho)$ *with* $\rho > 0$, *then* $a_n = b_n$ *for all* $n = 0, 1, 2, \cdots$.

(b) If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$$

for all $x \in (x_0 - \rho, x_0 + \rho)$ *with* $\rho > 0$, *then* $a_n = 0$ *for all* $n = 0, 1, 2, \cdots$.

- Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Rewrite the following expressions as the power series containing x^n.

1. $x^2 y'' - y$
2. $x^2 y'' + xy'$
3. $xy'' - 2y$
4. $y'' - 2y'$
5. $y'' + y$