8 The Laplace Transforms

Department of Mathematics & Statistics

ASU

1. Introduction to the L. T.
2. The Inverse of L. T.
3. Solution of IVPs
4. The Unit Step function
5. Constant Coefficient Eqs. with Piecewise continuous forcing functions
6. Convolution
7. Constant Coefficient Eqs with Impulses.

- Many practical engineering problems do not always require solutions which are continuous. The Laplace Transform is helpful to handle those problems mathematically.

- **Improper integrals** are defined over **the unbounded** domain:

$$\int_a^{\infty} g(t)\,dt = \lim_{T\to\infty}\int_a^{T} g(t)\,dt.$$

1. **Their convergence**: if the integrals from a to T exist for each $T > a$ and the limit exists as $T \to \infty$.
2. **Their divergence:** otherwise.

- Integral Transforms are very useful tools for solving linear DEs. An integral transform is a relation of the form

$$F(s) := \int_a^b K(s,t) f(t)\,dt, \tag{1}$$

where $K(s,t)$ is called the **kernel** of the transformation. For the Laplace transform the kernel $K(s,t)$ will be defined by

$$K(s,t) = e^{-st}.$$

Definition

Let f be defined for $t \geq 0$ and $s \in \mathbb{R}$ be fixed. Then the Laplace transform, denoted by $\mathscr{L}(f(t))$ is define by

$$\mathscr{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{2}$$

whenever this improper integral converges for appropriate $s \in \mathbb{R}$.

Theorem

(Shifting Theorem) If

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{for } s > s_0,$$

then $F(s-a) = \mathscr{L}\left(e^{at} f(t)\right)$ *for* $s > s_0 + a$.

- See the Laplace transform table on pp. 463–464.
- The Laplace transform is a linear operator. Assume that f_1, f_2 are two functions whose Laplace transform exist for $s > a_1$ and $s > a_2$. Then $s > \max\{a_1, a_2\}$ we have

$$\mathscr{L}\left(c_1 f_1(t) + c_2 f_2(t)\right) = c_1 \mathscr{L}\left(f_1(t)\right) + c_2 \mathscr{L}\left(f_2(t)\right).$$

- $f(t_0+) \neq f(t_0-)$ and they have finite values $\Rightarrow$ f has a **jump discontinuity** at $t = t_0$.
- $f(t_0) \neq f(t_0+) = f(t_0-)$ $\Rightarrow$ f has a **removable discontinuity** at $t = t_0$
- A function f is said to be **piecewise continuous** on the close interval $[a, b]$ if it is continuous there except for a **finite** number of **jump or removable discontinuities.**

Definition

A function f is said to be of exponential order s_0 if $\exists\, M, t_0$ such that

$$|f(t)| \leq Me^{s_0 t} \quad t \geq t_0.$$

Theorem

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $\mathscr{L}(f)$ is defined for $s > s_0$.