8.2 The Inverse Laplace Transform

- Recall the definition of the Laplace transform:

$$F(s) = \mathscr{L}(f) = \int_0^\infty e^{-st} f(t) t dt.$$

- Since $\mathscr{L}$ is an injective (1–1) operator, we can also an inverse Laplace transform

$$f = \mathscr{L}^{-1}(F).$$

Theorem

(Linear Property) *Let $c_1, c_2, \cdots, c_n$ be constants. Then*

$$\begin{aligned} &\mathscr{L}^{-1}(c_1 F_1 + c_2 F_2 + \cdots + c_n F_n) \\ &= c_1 \mathscr{L}^{-1}(F_1) + c_2 \mathscr{L}^{-1}(F_2) + \cdots + c_n \mathscr{L}^{-1}(F_n). \end{aligned}$$

- **How to write Partial Fractions for improper rational functions, i.e.,**

$$\frac{P(s)}{Q(s)} \text{ with } \mathbf{deg}(P) \leq \mathbf{deg}(Q).$$

(1) If $Q(s) = (a_1 s + b_1)(a_2 s + b_2)\cdots(a_n s + b_n)$ with deg $(Q) = n$, then there exist constants $A_1, A_2, \cdots, A_n$ such that

$$\frac{P(s)}{Q(s)} = \frac{A_1}{(a_1 s + b_1)} + \frac{A_2}{(a_2 s + b_2)} + \cdots + \frac{A_n}{(a_n s + b_n)}.$$

Note that $Q(s)$ does not have repeated factors and no factor is constant multiple of another.

(2) Suppose that $Q(s) = (a_1 s + b_1)(a_2 s + b_2)\cdots(a_n s + b_n)$ but some of the factors are repeated r times. For example, if $a_1 s + b_1$ is repeated r times, then we can use

$$\frac{A_1}{(a_1 s + b_1)} + \frac{A_2}{(a_1 s + b_1)^2} + \cdots + \frac{A_r}{(a_1 s + b_1)^r}.$$

(3) Suppose that Q contains **irreducible** quadratic factors, none of which is repeated. Then there exist constants $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ such that

$$\frac{P(s)}{Q(s)} = \frac{A_1 s + B_1}{(a_1 s^2 + b_1 s + c_1)} + \frac{A_2 s + B_2}{(a_2 s^2 + b_2 s + c_2)} + \cdots + \frac{A_n s + B_n}{(a_n s^2 + b_n s + c_n)}.$$

(4) Suppose that Q contains some **repeated irreducible** quadratic factors. For example, if $\left(a_n s^2 + b_n s + c_n\right)$ is repeated r times, then we can use

$$\frac{A_1 s + B_1}{(a_n s^2 + b_n s + c_n)} + \frac{A_2 s + B_2}{(a_n s^2 + b_n s + c_n)^2} + \cdots + \frac{A_r s + B_r}{(a_n s^2 + b_n s + c_n)^r}.$$