8.5 Constant Coefficient equations with piecewise continuous forcing functions

Theorem

Suppose that f is piecewise continuous on $[0, \infty)$ with jump discontinuities at $t_1, t_2, \cdots, t_n$ where $0 < t_1 < \cdots < t_n$. Let k_0 and k_1 be arbitrary real numbers. There $\exists!$ function y defined on $[0, \infty)$ satisfying the following conditions

(a) $y(0) = k_0$ and $y'(0) = k_1$

(b) y and y' are continuous on $[0, \infty)$.

(c) y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \cdots, t_n$ and

$$ay'' + by' + cy = f(t)$$

on every such interval, where $a \neq 0$, b, c are constants.

(d) y'' has limits from right and left at $t_1, \cdots, t_n$.

- How to solve the following IVPs

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases} \qquad y(0) = k_0, \quad y'(0) = k_1.$$

(1) Method#1
(step1) Find the solution y_0 on $[0, t_1)$ satisfying

$$ay'' + by' + cy = f_0(t) \quad y(0) = k_0, \quad y'(0) = k_1.$$

(step2) Compute $c_0 = y_0(t_1)$ and $c_1 = y_0'(t_1)$.
(step3) Find the solution y_1 on $[t_1, \infty)$ satisfying

$$ay'' + by' + cy = f_1(t) \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

(step4) Obtain the solution y as

$$y = \begin{cases} y_0(t), & 0 \le t < t_1, \\ y_1(t), & t \ge t_1. \end{cases}$$

(2) Method#2: Use Laplace transforms.