8.6 Convolution

- A Laplace transform $H(s)$ can be the product of two transforms $F(s)$ and $G(s)$, where $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$.
- You need to be very careful about the product. See the following Theorem below.

Theorem

Suppose that $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$ exist for $s > a \geq 0$, then

$$\mathscr{L}\{h(t)\} := H(s) = F(s)\,G(s) = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}, \quad \text{for } s > a,$$

where

$$h(t) := (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \quad (1)$$

The function h is known as the convolution of f and g.

- The equality of the two integrals in (1) can be proved by making the change of variable $\xi = t - \tau$.
- According to our observation of the previous theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms.
- Here are the properties of the convolution $f * g$.

1. Commutative law: $f * g = g * f$
2. Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. Associative law: $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$ but $(f * 1) \neq f$.

Theorem

Suppose that f is continuous on $[0, \infty)$ and $\exists \mathscr{L}(f)$. The the solution of the IVP

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + \int_0^t w(t) f(t - \tau) d\tau,$$

where y_1 and y_2 satisfy

$$ay_1'' + by_1' + cy_1 = f(t), \quad y_1(0) = 1, \quad y_1'(0) = 0,$$

and

$$ay_2'' + by_2' + cy_2 = f(t), \quad y_2(0) = 0, \quad y_2'(0) = 1,$$

and

$$w(t) = \frac{1}{a} y_2(t).$$