8.7 Constant Coefficient Equations with Impulses (Dirac Delta Function)

- When we work on engineering problems such as voltages or forces of large magnitude, we see that it is necessary to deal with phenomena of the impulsive natures. How do we formulate DEs (or PDEs) describing the physical phenomena? An impulse (the Dirac delta) function can be applied to set up the DEs. Such problems lead to the second order DEs

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is considerably large during a short time period $(t_0 - \tau,\, t_0 + \tau)$ and $g(t) = 0$ if $t \notin (t_0 - \tau,\, t_0 + \tau)$. So the function changes drastically over the time period.

- How do we justify such a function mathematically?

- For our simplicity, we assume that $t_0 = 0$. Then the integral function $I(\tau)$ is defined by:

$$I(\tau) = \int_{-\tau}^{\tau} g(t)dt.$$

In mechanical systems, $g(t)$ can be a force and $I(\tau)$ is called the total impulse of the force $g(t)$ over the interval $(-\tau, \tau)$.

- Assume that $g(t)$ is given by

$$g(t) := d_\tau(t) = \begin{cases} \frac{1}{2\tau} & \text{if } -\tau < t < \tau, \\ 0 & \text{if } t \leq -\tau \text{ or } t \geq \tau, \end{cases}$$

where $\tau > 0$ is sufficiently small. Now we pass the limit of τ and thus we can see that

$$\lim_{\tau \to 0} d_\tau(t) = 0, \quad \text{for } t \neq 0. \tag{1}$$

However, the limit of the total impulse I becomes

$$\lim_{\tau \to 0} I(\tau) = \lim_{\tau \to 0} \int_{-\tau}^{\tau} d_\tau(t)dt = 1. \tag{2}$$

- The equations (1) and (2) provide a good idea to define an idealized **unit impulse (Dirac delta) function** δ:

$$\delta(t) = \begin{cases} \infty & \textbf{if } t = 0, \\ 0 & \textbf{if } t \neq 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

By the shift of t_0, we can extend the Dirac delta function at everywhere $t = t_0$;

$$\delta\left(t - t_0\right) = \begin{cases} \infty & \text{if } t = t_0, \\ 0 & \text{if } t \neq t_0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta\left(t - t_0\right) dt = 1.$$

- Although δ is not bounded, the Laplace transform of δ can be defined. For any $t_0 > 0$ we can define

$$\mathscr{L}\left\{\delta\left(t - t_0\right)\right\} = \mathscr{L}\left\{\lim_{\tau \to 0} d_\tau\left(t - t_0\right)\right\}.$$

Since the Laplace transform is a linear operator, we can have

$$\mathscr{L}\left\{\lim_{\tau \to 0} d_\tau\left(t - t_0\right)\right\} = \lim_{\tau \to 0} \mathscr{L}\left\{d_\tau\left(t - t_0\right)\right\}.$$

- We can see that $\mathscr{L}\left\{\delta\left(t-t_0\right)\right\}=e^{-st_0}$ and $\mathscr{L}\left\{\delta\left(t\right)\right\}=1$ as $t\to 0$.
- In a similar way, we can define the inner product of the Dirac delta function and any **continuous** function f:

$$\int_{-\infty}^{\infty}\delta\left(t-t_0\right)f(t)dt=\lim_{\tau\to 0}\int_{-\infty}^{\infty}d_\tau\left(t-t_0\right)f(t)dt=f\left(t_0\right).$$

Definition

(pp. 458) Suppose that α is a nonzero constant and f is piecewise continuous on $[0, \infty)$. If $t_0 > 0$, then the solution of the IVP

$$ay'' + by' + cy = f(t) + \alpha\delta(t - t_0), \quad y(0) = k_0, \quad y'(0) = k_1$$

is defined to be

$$y(t) = \hat{y}(t) + \alpha u(t - t_0)\, w(t - t_0),$$

where $\hat{y}$ is the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

and

$$w(t) = \mathscr{L}^{-1}\left(\frac{1}{as^2 + bs + c}\right).$$

This definition also applies if $t_0 = 0$, provided that the initial condition $y'(0) = k_1$ is replaced by $y'_-(0) = k_1$.

Examples

- Solve the following IVPs:

1. $$y'' + y = \delta(t-1), \quad y(0) = y'(0) = 0.$$

2. $$y'' - 2y' - 3y = \delta(t-1), \quad y(0) = y'(0) = 0.$$

3. $$y'' - 2y' + y = 3\delta(t-2), \quad y(0) = y'(0) = 0.$$