9.2 Higher Order Constant Coefficient Homogeneous Eqs.

- The nth order linear homogeneous DE with constants a_i for $i = 0, 1, 2, \cdots, n$ can be written by

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

By the similar idea of what we did for the second order DEs, we can set up the characteristic Eq.

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

- Unlike the second order DEs, we cannot find solutions for the Eqs. of nth order in general. There are some ways such as the **synthetic division** or **complex analysis** to solve the characteristic Eqs.

- Depending on the type of solutions of $Z(r) = 0$, we consider the following cases.

1. **Solutions** r **are real and nonrepeated**, i.e., $r = r_i \in \mathbb{R}$ for $i = 1, 2, \cdots, n$. Then the general solution becomes

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}.$$

2. **Solutions** r **are complex and nonrepeated**, i.e., $r = \lambda_j \pm i\,\mu_j$. Then the general solution becomes

$$y = e^{\lambda_1 t}(c_1 \cos \mu_1 t + d_1 \sin \mu_1 t) + \cdots + e^{\lambda_j t}(c_j \cos \mu_j t + d_j \sin \mu_j t) + \cdots$$

The general solution seems to be unclear. We will understand this case completely, by doing examples.

3. **Solutions** r **are repeated.**
(1) If the real root r is repeated s times, then the general solution becomes

$$y = c_1 e^{rt} + c_2 t e^{rt} + c_3 t^2 e^{rt} + \cdots + c_s t^{s-1} e^{rt}.$$

(2) If $r = \lambda \pm i\,\mu \in \mathbb{C}$ and the complex root is repeated s times, the general solution becomes

$$y = e^{\lambda t}(c_1 \cos \mu t + d_1 \sin \mu t) + e^{\lambda t} t (c_2 \cos \mu t + d_2 \sin \mu t) + \cdots + e^{\lambda t} t^{s-1} (c_s \cos \mu t + d_s \sin \mu t).$$

- How to find solutions of the following Eq.: for any $a \in \mathbb{R}$

$$r^n = a \quad \text{with } n \geq 1. \tag{1}$$

1. Switch a into the polar form:

$$a = |a|(\cos(2m\pi + \theta) + i \sin(2m\pi + \theta)) = |a| e^{i(2m\pi+\theta)}.$$

2. From the original Eq. (1) we take $n$the radical root. Then for $m = 0, 1, 2, \cdots, n-1$ we have the following solutions

$$r = |a|^{1/n} e^{i(2m\pi+\theta)/n} = |a|^{1/n} \left(\cos\left(\frac{2m\pi+\theta}{n}\right) + i \sin\left(\frac{2m\pi+\theta}{n}\right)\right).$$