1. The Foundations: Logic and Proofs

Department of Mathematics & Statistics

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Why do we study D. S.?

- **Discrete Structures (Mathematics)**

1. is to study mathematical structures (theories) that are fundamentally discrete rather than continuous.
2. So Calculus and Analysis is not included in D.M.
3. The set of elements studied in D.M. can be finite or (countably) infinite. Finite Mathematics is a part of D.M.

- **Discrete vs Continuous**

1. We can understand their difference, considering a number system; for example,
   Natural (Whole) numbers for the discrete case
   Real numbers for the continuous case.
2. **Discretization** is related to the process to transferring continuous equations into corresponding discrete formulas. **Numerical analysis** is an important example of discretization, because the continuous equations are discretized by using approximations.

Outline of Chapter 1

1. Propositional Logic
2. Applications of Propositional Logic
3. Propositional Equivalences
4. Predicates and Quantifiers
5. Nested Quantifiers
6. Rules of Inference
7. Introduction to Proofs
8. Proof Methods and Strategy

- Mathematical proofs are important in computer science, because proofs are used to show that algorithms generated by mathematical ideas produce the correct result, to establish the security of a system, and to created artificial intelligence (AI).

1.1 Propositional Logic

- A major aim in this course is to learn how to understand and construct mathematical arguments. In order to do so, we better begin with an introduction of **logic**. Logic has many applications to computer science.
- The basic building blocks of logic is a **proposition** that is a declarative sentence. When we can decide if a sentence is either true or false, but not both, it is called a proposition.
- The truth value of a proposition is denoted by T if it is true. The truth value of a proposition is denoted by F if it is false.

Example1

Which of these sentences are propositions? What are the truth values of those that are propositions?

1. Little Rock is the capital city of Arkansas.
2. Dr. Ahn is a well-known mathematician.
3. $1+1=1$
4. $x-3=2$.

- We use lowercase letters $p, q, r, s, \cdots$ to denote propositional variables (or statement variables).
- New propositions, called **compound propositions**, are formed from existing propositions using **logical operators**.
- There are several logical operators (**1.negation 2.and 3.or**), called connectives.
- Let p and q be propositions.

Definition

The **negation** of p, denoted by $\neg p$(or $\sim p$,) is the statement

$$\text{"It is not the case that } p\text{."}$$

The $\neg p$ is read "not p". The truth value of $\neg p$ is the opposite of the truth value of p.

Example2

Find the negation of the proposition

"Jacob's iphone has at least 16GB of memory."

Definitions

1. The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p **and** q".
2. The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p **or** q".
3. The **exclusive or of** p and q, denoted by $p \oplus q$, is the proposition that is T when exactly one of p and q is T and is F otherwise.

Conditional Statement (Implication)

Definitions

The conditional statement of $p \to q$ is the proposition "if p, then q." p is called the **hypothesis** and q is called the **conclusion**.

- We can use more ways to express conditional statements: (1) if p, q (2) p is sufficient for q (3) q if p (4) q when p (5) a necessary condition for p is q (6) q is unless $\neg p$.
- There are several new conditional statements with an original conditional statement $p \to q$.

1. $q \to p$ is the **converse** of $p \to q$.
2. $\sim q \to \sim p$ is the **contrapositive** of $p \to q$.
3. $\sim p \to \sim q$ is the **inverse** of $p \to q$.

Definition

The **biconditional statement** $p \leftrightarrow q$ is the proposition "p if and only if q".

The Truth Table

- Negation, Conjunction, Disjunction, and Exclusive

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

- Conditional and biconditional statements

p	q	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

- **Precedence of Logical Operators**

1. The very first step is that we need to work inside of **parentheses**.
2. Precedence of Logical Operators:
   1. $\sim$ **2.**$\wedge, \vee$ **3.** $\rightarrow$ **4.** $\leftrightarrow$.

- **Logic and Bit Operations**

1. A bit is a symbol with two possible values, namely, 0 and 1. Computer bit operations corresponding to the logical connectives.

True Value	Bit
T	1
F	0

2. Using bit rather than True value gives easier way to remember the truth tables in the previous page.

Examples

Example3

Construct the truth table for each of these compound propositions.

1. $p \wedge \sim p$
2. $(p \vee \sim q) \rightarrow (p \wedge q)$
3. $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
4. $(p \oplus q) \leftrightarrow (p \oplus \sim q)$
5. $(\sim p \leftrightarrow \sim q) \leftrightarrow (q \leftrightarrow r)$

Example4

p :It is below freezing, q :It is snowing. Then write these proposition using $p, q, \sim$.

1. It is below freezing and snowing.
2. If it is below freezing, then it is snowing.
3. That it is below freezing is necessary and sufficient for it to be snowing.