

1.4 Predicates & Quantifiers

- **Propositional logic** cannot sufficiently express the whole meaning of all statements in the mathematics and natural language. **Predicate logic** is more powerful and allows us to avoid confusions of Propositional logic.
- Consider a mathematical example: $x < 2$.
 x is called a variable and the corresponding predicate is “is less than 2”. The predicate is denoted by P . So we write $P(x) : x < 2$, where the statement $P(x)$ is said to be the value of the **Proposit. func. P** at x . Thus, $P(x)$ is always T or F .
- Propositional functions P may have more than one variable, using $P(x, y)$. In general, we may think of $P(x_1, x_2, \dots, x_n)$.

Examples

1. Let $P(x)$ denote the statement $x > 5$. What are the truth values of $P(1)$ and $P(8)$?
2. $Q(x, y) : x = y + 2$. Tell the truth values of $Q(3, 1)$ and $Q(1, 2)$?
3. $P(x_1, x_2, x_3) : x_1^2 + x_2^2 = x_3^2$. What is the truth value of $P(2, 3, 5)$?

- **Quantification** which is another important way to create a proposition from a propositional function expresses the extent to which a predicate is true over a range of elements. *all, some, many, none, and few* are called quantifiers.
- There are two types of quantification:
 - 1 universal quantification: $P(x)$ is true for every element x under consideration
 - 2 existential quantification: there is one or more element x under consideration for which $P(x)$ is true.
- The area of the logic that deals with predicates and quantifiers is called the **predicate calculus**.
- **Domain:** a set of all possible values of a variable.

Definitions

1. The **universal quantification** of $P(x)$ is the statement " $P(x)$ for **all** values of x in the domain". Its notation is $\forall xP(x)$, where \forall is called the **universal quantifier**. An element for which $P(x)$ is false is called a **counterexample** of $\forall xP(x)$.
2. The **existential quantification** of $P(x)$ is the proposition "There **exists** an element x in the domain such that $P(x)$ ". Its notation is $\exists xP(x)$, where \exists is called the **existential quantifier**.

- The meaning of the quantifiers are summarized in the table.

Statement	When True(T)?	When False(F)?
$\forall xP(x)$	$P(x)$ is T for every x	$\exists x$ for which $P(x)$ is F
$\exists xP(x)$	$\exists x$ for which $P(x)$ is T	$P(x)$ is F for every x

Example 1

1. $P(x) : x^2 \geq 0$. What is the truth value of $\forall x P(x)$ for the domain $D = (-\infty, \infty)$?
2. $Q(x) : x < -1$. What is the truth value of $\forall x Q(x)$ for the domain $D = (-\infty, \infty)$?
3. $R(x) : x^2 > 0$. What is the truth value of $\forall x R(x)$ for the domain $D = (-\infty, \infty)$?
4. $P(x) : x > 2$. What is the truth value of $\exists x P(x)$ for the domain $D = (-\infty, \infty)$?
5. $Q(x) : x = x + 2$. What is the truth value of $\exists x P(x)$ for the domain $D = (-\infty, \infty)$?

Examples

- **U. Q.:** for all = for every = all of = for each = given any = for arbitrary.
E. Q.: there exists = for some = for at least one = there is.
- For countable finite elements x_1, x_2, \dots, x_n in the domain:
 1. $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$
 2. $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$

Example2

Let the domain $D = \{x \in \mathbb{Z} \mid 0 \leq x \leq 3\}$ with the set of integers \mathbb{Z} .

1. What is the truth value of $\forall x P(x)$, where $P(x) : x^2 < 4$?
2. What is the truth value of $\exists x P(x)$, where $P(x) : x^2 < 4$?

Negating Quantified Expressions

- De Morgan's Laws for Quantifiers

Negation	E. S.	When True?	When is False?
$\sim \exists x P(x)$	$\forall x \sim P(x)$	For all x , $P(x)$ F	$\exists x$ for wh. $P(x)$ T
$\sim \forall x P(x)$	$\exists x \sim P(x)$	$\exists x$ for wh. $P(x)$ F	For all x , $P(x)$ T

- De Morgan's laws can be similarly applied into countable finite elements x_1, x_2, \dots, x_n in the domain.

Example3

- What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?
- Suppose that the domain of $P(x)$ is $D = \{0, 1, 2, 3, 4\}$. Write out followings, using disjunction, conjunction, and negation.
(1) $\exists x P(x)$ (2) $\exists x \sim P(x)$ (3) $\sim \forall x P(x)$ (4) $\sim \exists x P(x)$

Translating from English to Logical Expressions

- Translating mathematical or English sentences into logical expressions is a crucial task in mathematics, logic programming, artificial intelligence, software engineering,

Example4

1. Express the statement “All students in the class study elementary calculus” using predicates and quantifiers.
2. Express the statements “Some student in this class visits Mexico” and “every student in this class visits either Mexico or Canada” using predicates and quantifiers.