1.5 Nested Quantifiers (N. Qs)

- N. Qs are nested if one Q is within the scope of another.
  Example: $\forall x \exists y\, (x + y = 1)$ can be easily understood as

$$\forall x\, Q(x)$$
$$Q(x) \text{ is } \exists y\, P(x, y)$$
$$P(x, y) \text{ is } (x + y = 1)$$

Example1

Translate into English the following statements. $D = \mathbb{R}$.
(1) $\forall x \forall y\, (x + y = y + x)$. (2) $\forall x \forall y\, ((x > 0) \wedge (y > 0) \to (x\, y > 0))$.

Example2

Determine the truth value of the following nested quantifications, $D = \mathbb{Z}$.
(1) $\forall n \exists m\, (n^2 \leq m)$ (2) $\exists n \forall m\, (n\, m = m)$ (3) $\exists n \exists m\, (n^2 + m^2 = 5)$
(4) $\forall n \forall m \exists p\, (p = (m + n)/3)$

The Order of Universal (Existential) Quantifiers

Example3

1. Let $P(x,y): x+y=y+x$. Tell the truth values of $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$, where $D=\mathbb{R}$.
2. Let $P(x,y): x+y=2$. Tell the truth values of $\exists x \exists y P(x,y)$ and $\exists y \exists x P(x,y)$, where $D=\mathbb{Z}$.

- From the previous example, we can see that even if the order of nested **universal (existential)** quantifiers in a statement is changed, the meaning of the quantified statements will be the same.

The Order of mixtures of U. and E. Quantifiers

- The order of nested existential and universal quantifiers in a statement has to be considered with more careful treatment.

Example4

1. Let $Q(x,y): x+y=1$. Tell the truth values of $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$, where $D=\mathbb{R}$.
2. Let $P(x,y,z): x+y=z$. Tell the truth values of $\forall x \forall y \exists z\, Q(x,y,z)$ and $\exists z \forall x \forall y P(x,y,z)$, where $D=\mathbb{R}$.

- As we did in the previous example, switching the order of nested existential and universal quantifiers in a statement may make a difference. But switching the order may not make a difference. Here is an example.

Example5

Let $Q(x,y): xy=1$. Tell the truth values of $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$, where $D=\mathbb{R}$.

Quantifications of two variables

- The following table will summarize the meanings of different possible quantifications.

Statement	When True(T)?	When False(F)?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$: T for every pair x,y	There is a pair x,y for which $P(x,y)$: F
$\forall x \exists y P(x,y)$	For every x there is y for which $P(x,y)$: T	There is an x such that $P(x,y)$: F for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$: T for every y	For every x there is a y for which $P(x,y)$: F
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which $P(x,y)$: T	$P(x,y)$: F for every pair x,y

- From Nested Quantifiers into English

Example6

1 .Let $D = \{(x, y) \mid x, y$ are students in ASU$\}$ be the domain.
Then, translate into English the statement
$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$,
where $C(x)$: "x has a computer", $F(x, y)$ is "x and y are friends".
2. $Q(x, y)$:x has sent an email message to y, where
$D = \{(x, y)|x, y$ are students in ASU$\}$. Express the followings in English. (1) $\exists x \exists y Q(x, y)$ (2) $\forall y \exists x Q(x, y)$
3. Translate the following nested quantifications into English, where $D = \mathbb{R}$.
(1) $\exists x \forall y(xy = y)$ (2) $\forall x \forall y((x < 0) \wedge (y < 0)) \rightarrow (x y > 0))$

- From English into Nested Quantifiers

Example7

Express the following statements as a logical expression involving predicates, quantifiers, and logical connectives.
(1) Everyone has exactly one best friend, D is a set of all people.
(2) All math and computer science students need to take Discrete Structures. D is a set of C.S and math students.
(3) Every student in our class has taken at least one mathematics course.
$D = \{(x, y) | x$ is a student in our class and y is a math course$\}$.