1.7 Introduction to Proofs

- A proof: a valid argument that establishes the truth of a mathematical statement.
- In order to prove lemmas or theorems, the hypotheses of the theorems, axioms (postulates), definitions or previously proven theorems can be used.
- Some Terminologies

  1. **Theorem:** a statement that can be shown to be true. A less important theorem is called a **proposition** sometimes.

  2. **Lemma:** a less important theorem that is helpful in the proof of other results

  3. **Corollary:** a theorem that can be established directly from a theorem that has been proved.

  4. **Conjecture:** a statement that is being proposed to be a true statement. When a proof is correct, the conjecture becomes a theorem.

- **Direct Methods:** show that $p \to q$ is true by showing that if p is true, then q must also be true. So in order to show that q must also be true, we assume that p is true, and use axioms, definitions, and previously proven theorem, together with rules of inference.

Definitions

Recall the number system $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. Then,
1. the integer n is even if $\exists k \in \mathbb{Z}$ such that $n = 2k$.
2. the integer n is odd if $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$.

Example1

1. Use a direct proof to show that if n is an odd integer, then n^2 is odd.
2. Use a direct proof to show that the sum of two odd integers is even.

- **Contraposition:**
  Proofs that do not start with the premises and do not end with the conclusion are called **indirect proofs.** A useful type of the indirect proofs is known as **proof by contraposition,** which is based on the equivalence that $p \to q \iff \sim q \to \sim p$.

Example2

1. Use a proof by contraposition to show that if $n \in \mathbb{Z}$ and $3n+2$ is odd, then n is odd.
2. Use a proof by contraposition to show that if $x+y \geq 2$, then $x \geq 1$ or $y \geq 1$, where $x, y \in \mathbb{R}$.

- **Vacuous (or Trivial) Proofs**
  If we can show that an assumption is false, we will be done.

Example3

Show that $P(0)$ is true, where $P(n)$: if $n > 1$, then $n^2 > n$ for $n \in \mathbb{Z}$.

Indirect Proofs

- **Contradiction:**
  1. If there is **one proposition** p, take a negation of p and then find a contradiction. This idea is based on the fact that if p is true, then $\sim p \to (r \wedge \sim r)$ is true.
  2. For the conditional statement $p \to q$, assume that p and $\sim q$ is true. If we show that both $\sim p$ are true, we have a contradiction $p \wedge \sim p$. Simply Speaking, consider a negation of q and then find a contradiction of the assumption.
- **Proofs of equivalence:**
  To prove a theorem that has $p \leftrightarrow q$, show that $p \to q$ and $q \to p$ are both true.

Example4

1. Use a proof by contradiction to show that (1) $\sqrt{3} \notin \mathbb{Q}$ (2) the sum of an irrational number and rational number is irrational.
2. Prove that if $x \notin \mathbb{Q}$, then $1/x \notin \mathbb{Q}$.
3. Assume that $1 \leq n \in \mathbb{Z}$. Show that n is even $\Leftrightarrow 5n + 6$ is even.