5.1 Mathematical Induction

- Imagine climbing an infinite ladder, reaching every its rung. That will be helpful to understand mathematical induction.
- A proof by mathematical induction will be carried out, based on two parts: a basis step and inductive step

Definition

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n. we go through the following two steps.
1. **Basis step**: Verify that $P(1)$ is true.
2. **Inductive step**: Show that $P(k) \to P(k+1)$ is true for $\mathbb{Z} \ni k \geq 1$.

The assump. that $P(k)$ is true is called the **inductive hypothesis**.

- The good: M. I. can be used to prove a conjecture, once the conjecture has been made (and is true).
  The bad: M. I. cannot be used to establish new conjectures directly.

Guidelines for Proofs by Mathematical Induction

1. Express the mathematical statement that is to be proved in the form "for all $n \geq b$, $P(n)$, for a fixed $b \in \mathbb{Z}$.
2. Write out the words **"Basis Step"**. Then show that $P(b)$ is true.
3. Write out the words **"Inductive Step"**.
4. State and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$.
5. Write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$, using the assumption $P(k)$.
7. Clearly identify the conclusion of the inductive step, saying "this completes the inductive step."
8. State the conclusion, saying "by mathematical induction, $P(n)$ is true for all integer $n \geq b$".

Examples

Example1

1.Show that for all integer $n \geq 1$, then

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

2. Show that or all integer $n \geq 1$, then

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. **Geometric sums**: Prove the following formula

$$\sum_{j=0}^{n} a\, r^j = a + ar + ar^2 + \cdots + ar^n = \frac{a\left(r^{n+1}-1\right)}{r-1} \quad \text{when } r \neq 1,$$

where a is an initial term and r is the common ratio.

Cont'd

Example2

1.Prove that for $n \in \mathbb{Z}^+$

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}.$$

2. Prove that for all integers $n \geq 1$

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

3. Prove that $n^2 \leq 2^n$ for all integers $n \geq 4$.
4. Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.