2.3 Functions

- In modern mathematics, we cannot develop mathematical theories without understanding functions (or mappings, or transformation, or operators)

Definitions

A **function** $f : A \to B$ relates each element of a set A with **exactly one** element of another set B.
1. We write $b = f(a)$ if $\exists! b \in B$ assigned by f to the element $a \in A$.
2. A: the domain of the function f and B: the codomain of f and the set of all images is called the range included in B.

- The vertical line test is useful to determine whether graphs are functions or not.

Example1

Determine if the following $f : \mathbb{R} \to \mathbb{R}$ is a function.
(1) $f(x) = 1/x^2$ (2) $f(x) = \sqrt{x}$ (3) $f(x) = \pm\sqrt{-x^2+1}$.

Definitions

1. A function $f : A \to B$ is said to be 1-1 or injective if $f(a) = f(b)$ implies that $a = b$ for all $a, b \in A$.
2. A function $f : A \to B$ is said to be onto or surjective if for all $c \in B$, $\exists a \in A$ such that $f(a) = c$.

Example2

1. Determine if the followings $f : \mathbb{Z} \to \mathbb{Z}$ are 1-1 or onto.
(1) $f(n) = n + 1$ (2) $f(n) = n^3$ (3) $f(n) = \lceil n/2 \rceil$.
2. Determine if the followings $f : \mathbb{R} \to \mathbb{R}$ are 1-1 or onto.
(1) $f(x) = 2x - 1$ (2) $f(x) = x^2$ (3) $f(x) = \left(x^2 + 1\right) / \left(x^2 + 2\right)$.

Definitions

Let f and g be functions from A to $\mathbb{R}$.
1. For operators $+$ and $\cdot$, $f+g$ and fg are functions from A to $\mathbb{R}$ defined for all $x \in A$ by

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x).$$

2. For the operator $\circ$, the composition of $g : A \to B$ and $f : B \to C$, denoted by $f \circ g$ is defined for all $x \in A$ by

$$(f \circ g)(x) = f(g(x)).$$

Example3

Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = x^2 + 1$ and $g(x) = 3x + 2$. Then find the following: (1) $f+g$ (2) fg (3) $f \circ g$ (4) $g+f$ (5) gf (6) $g \circ f$.

Definitions

1. The **floor function** assigns to $x \in \mathbb{R}$ the **largest integer** that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor \in \mathbb{Z}$.
2. The **ceiling function** assigns to $x \in \mathbb{R}$ the **smallest integer** that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil \in \mathbb{Z}$.
3. Let $f : A \to B$ and a function and let $S \subset A$. Then the image of S is denoted by $f(S)$, i.e.,

$$f(S) = \{f(s) \mid s \in S\}.$$

Example4

1. Let $f(x) = \lceil x^2/3 \rceil$. Find $f(S)$ if (1) $S = \{-2, -1, 0, 1, 2, 3\}$ and (2) $S = \{0, 1, 2, 3, 4, 5\}$
2. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if (1) $S = \{1, 5, 7, 11\}$ and (2) $S = \{2, 6, 10.14\}$.