- **Sequences vs Sets**
  Sequences are **ordered** lists of elements, while sets are collections of elements regardless of order.
- A finite sequence: $\{a_1, a_2, \cdots, a_n\}$ (called a string) and an infinite sequence: $\{a_1, a_2, \cdots, a_n, \cdots\}$.

Definition

A sequence is a **function** f from $A \subset \mathbb{Z}$ to a set S. So the image of the integer n is denoted by $a_n = f(n)$. a_n is called a **general term** of the sequence. The sequence is denoted by $\{a_n\}$.

Example1

Write the first five terms of the sequence $\{a_n\}$ defined by $a_n = (-1)^n/n$. for $n \geq 1$.

Definitions

1. A **geometric** progression is a sequence of the form $a_n = ar^{n-1}$ for $n \geq 1$, where a is called the initial term and r is the common ratio. So $r = a_{n+1}/a_n$.
2. An **arithmetic** progression is a sequence of the form $a_n = a + (n-1)d$ for $n \geq 1$, where a is called the initial term and d is the common difference. So $d = a_{n+1} - a_n$.

Example2

1. Let $a_n = 2 \cdot (-3)^n + 5^n$. Then find (1) a_0 (2) a_4.
2. Let $a_n = 2n + 1$. Then find (1) a_1 (2) a_3.
3. Let $a_n = (n+1)^{n+1}$. Then find (1) a_0 (2) a_2.
4. Let $a_n = \lfloor n/2 \rfloor$. Then find (1) a_1 (2) a_3.

Definition

A **recurrence relation** for $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms for $n \geq n_0 \geq 0$.

Example3

For $n \geq 1$

1. let $a_n = a_{n-1} + 3$. If $a_0 = 2$, then find (1) a_1 (2) a_3.
2. let $a_n = a_{n-1}^2$. If $a_0 = 2$, then find the next five terms.
3. let $a_{n+1} = a_n + a_{n-1}$. (Fibonacci sequence) If $a_0 = 0$ and $a_1 = 1$, then find the next five terms.

- We use the Σ notation to express the sum of finitely (infinitely) terms:

$$a_m + a_{m+1} + \cdots + a_n = \sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{k=m}^{n} a_k = \cdots,$$

where j is the index of summation and m is the lower limit and n is the upper limit.

Example4

What are the values of the following sums?
(1) $\sum_{k=1}^{5}(k+1)$ (2) $\sum_{i=1}^{10} 3$ (3) $\sum_{j=0}^{8}\left(2^{j+1}-2^{j}\right)$.

Example5

Compute the following double sums
(1) $\sum_{i=1}^{2}\sum_{j=1}^{3}(2i+3j)$ (2) $\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

- Some Useful Summation Formula

Sum	Closed form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	If $n \neq \infty$, $\frac{a\left(r^{n+1}-1\right)}{r-1}$ for$r \neq 1$, If $n = \infty$, $\frac{-a}{r-1}$ for $\lvert r \rvert < 1$.
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\left(\frac{n(n+1)}{2}\right)^2$

Example5

Find the following sums
1. $\sum_{k=5}^{10} k$ (2) $\sum_{i=3}^{5} k^3$

Example6

1. Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$.
This type of sum is called **telescoping sum.**
2. Find $\sum_{j=1}^{10} \frac{1}{j(j+1)}$. We need to use telescoping sum.