2.5 Cardinality of Sets

- It is easy to determine the cardinality of finite sets.
- How can we determine the cardinality of infinite sets? If we show that there exists bijection f from one set onto another, we can say that the two sets have the same cardinality.
- There are two types of infinite sets; one is a countably infinite set and another is a uncountably infinite set. For example,

1. Countable sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.
2. Uncountable sets: $\mathbb{R}, \mathbb{C}$.

Definition

Let $|A|$ be the cardinality of the set A. Then, $|A| = |B| \Leftrightarrow$ there is a one to one correspondence (bijection) from A onto B.

Definitions

1. **Countable sets** are finite sets or have the same cardinality as $\mathbb{N}$. When S is an infinite set, $|S| = \aleph_0$.
2. **Uncountable sets** are **not** countable sets.

Example1

1. Show that the set of positive even integers is a countable set.
2. Show that $\mathbb{Z}$ is a countable set.

- However, $\mathbb{R}$ is a uncountable set.
- According to the continuum hypothesis, there is no cardinal number between $\aleph_0$ and $|\mathbb{R}|$.

2.6 Matrices

- Matrices are used to solve linear systems.
- Matrices are used in models of communication networks and transportation systems and are used to compute approximations of PDEs.
- In this section, we go over matrix arithmetic.

Definitions

1. A matrix is a rectangular array of elements (numbers).
2. A matrix with m rows and n columns is called an $m \times n$ matrix.
3. If $m = n$, a matrix is called square.
4. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices. Then $\mathbf{A} = \mathbf{B}$ if they have the same number of lows and columns and the corresponding entries in every position are equal.

Definitions

Let **A** be a matrix with its size $m \times n$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-11} & a_{m-12} & \cdots & a_{m-1n-1} & a_{m-1n} \\ a_{m1} & a_{m2} & \cdots & a_{mn-1} & a_{mn} \end{bmatrix}.$$

The ith row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \cdots, a_{in-1}, a_{in}]$ and the jth column of **A** is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{m-1j} \\ a_{mj} \end{bmatrix}.$$

- From the def. the (i,j)th element or entry of $\mathbf{A}$ is the element a_{ij}, that is, the number in the ith row and jth column of $\mathbf{A}$.

Definition

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of $\mathbf{A}$, denoted by $\mathbf{A}^t$ is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. If a square matrix $\mathbf{A}^t = \mathbf{A}$, $\mathbf{A}$ is called symmetric.

Example1

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 7 \\ 9 & 1 & -3 \\ -2 & 8 & 0 \end{bmatrix}.$$

(1) What is the size of $\mathbf{A}$? (2) What is the third column of $\mathbf{A}$?
(3) What is the second row of $\mathbf{A}$? (4) What is $\mathbf{A}^t$?
(5) Find the element of $\mathbf{A}$ in the $(4,2)$th position.

Definitions

1. **Sum**+: Assume that **A** and **B** have the same size $m \times n$. Then the **sum** of **A** and **B**, denoted by $\mathbf{A}+\mathbf{B}$ is the $m \times n$ matrix that has $[a_{ij}+b_{ij}]$.
2. **Product**·: If **A** has the size $m \times k$ and **B** has the size $k \times n$, then the product of **A** and **B**, denoted by $\mathbf{A}\mathbf{B}$ is the $m \times n$ matrix. The method to find **AB** will be explained.

Example2

Let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & 4 \\ 5 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 4 & 3 \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} 1 & -2 \\ 6 & 5 \\ -7 & 0 \end{bmatrix}.$$

Then find **AB** and $\mathbf{A}+\mathbf{C}$.

Zero-one Matrices with Join, Meet, Product

- A matrix whose entries are either 0 or 1 is called a **zero-one matrix.** This matrix arithmetic is based on the Boolean operations $\wedge$ and $\vee$, which operate on pairs of bits. Thus, we can define

$$b_1 \wedge b_2 = \begin{cases} 1 \text{ if } b_1 = b_2 = 1, \\ 0 \quad \text{otherwise,} \end{cases} \quad b_1 \vee b_2 = \begin{cases} 1 \text{ if } b_1 = 1 \text{ or } b_2 = 1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Definitions

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero-one matrices. Then,
1. the join of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \vee \mathbf{B}$ is $a_{ij} \vee b_{ij}$
2. the meet of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \wedge \mathbf{B}$ is $a_{ij} \wedge b_{ij}$

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. Then the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$ is the $m \times n$ matrix with (i,j)th entry c_{ij}, where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$.

Definition

Let $\mathbf{A}$ be a square matrix and r be a positive integer. The rth Boolean power of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. Then rth Boolean product of $\mathbf{A}$, denoted by $\mathbf{A}^{[r]}$ is defined by $\mathbf{A}^{[r]} = \mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}$. We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

Example3

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

1. Find (1) $\mathbf{A} \vee \mathbf{B}$ (2) $\mathbf{A} \wedge \mathbf{B}$ (3) $\mathbf{A} \odot \mathbf{B}$.
2. Find (1) $\mathbf{A}^{[2]}$ (2) $\mathbf{A}^{[3]}$ (3) $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]}$.