3.2 Growth of Functions

- When we analyze some algorithms, we may want to consider the number of steps. For example, assume that the number of steps to complete a mathematical problem of size n is given by $f(n) = 3n^2 + 4n - 1$. If we ignore constants and slower growing terms, we can say that $f(n)$ depends on the dominating term n^2 for sufficiently large numbers n. How can we describe it, using a mathematical notation? The growth of functions can be written using big-O notation.

Definition

Let f and g be functions with the domain $\mathbb{R}$. We say that $f(x) = O(g(x))$ if there is a constant $C > 0$ and $k \in \mathbb{R}$ such that $|f(x)| \leq C|g(x)|$, provided that $x \geq k$. C and k are called witnesses.

- The growth of functions commonly used in Big-O estimates. Have a look at Figure 3 in page 211.

Theorem1

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{R}$ for $0 \le i \le n$. Then $P(x) = O(x^n)$.

- There are three important rules to find big-O notation.

Theorem2

1. Suppose that $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$. Then $(f_1(x) + f_2(x)) = O(\max(|g_1(x)|, |g_2(x)|))$.
2. Suppose that $f_1(x) = O(g(x))$ and $f_2(x) = O(g(x))$. Then $(f_1(x) + f_2(x)) = O(g(x))$.
3. Suppose that $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$. Then $(f_1(x) f_2(x)) = O(g_1(x) g_2(x))$.

Examples

Example

In problems 1-4, find witnesses C and k.

1. Let $f(x) = x^2 + 2x + 1$. Show that $f(x) = O\left(x^2\right)$.

2. Determine whether the followings can make $f(x) = O(x)$.

(1) $f(x) = 10^{10}x + 1$ (2) $f(x) = 10^{-6}x^2 + x + 2$ (3) $f(x) = 10\log x$

(4) $f(x) = \lfloor x \rfloor$ (5) $f(x) = \lceil x/2 \rceil$

3. Let $f(x) = \left(x^2 + 1\right)/(x + 1)$. Then show that $f(x) = O(x)$.

4. Find the least integer n such that $f(x) = O\left(x^n\right)$.

(1) $f(x) = 2x^3 + x^2\log x$ (2) $f(x) = 3x^3 + (\log x)^4$

(3) $f(x) = \left(x^4 + x^2 + 1\right)/\left(x^3 + 1\right)$

(4) $f(x) = \left(x^4 + 5\log x\right)/\left(x^4 + 1\right)$

5. Find the best corresponding big-O notation.

(1) $\left(n^2 + 8\right)(n + 1)$ (2) $\left(n\log n + n^2\right)\left(n^3 + 2\right)$

(3) $\left(n! + 2^n\right)\left(n^3 + \log\left(n^2 + 1\right)\right)$ (4) $n\log\left(n^2 + 1\right) + n^2\log n$

(5) $(n\log n + 1)^2 + (\log n + 1)\left(n^2 + 1\right)$ (6) $n^{2^n} + n^{n^2}$.