6.2 The Pigeonhole Principle

Theorem

The Pigeonhole Principle
If $k > 1$ is an integer and $k + 1$ or more objects (e.g., pigeons) are placed into k boxes (e.g., pigeonholes), then there is at least one box containing two or more of the objects.

- The pigeonhole principle can be proved, using a proof by contradiction.
- The principle can be applied to functions. See the next corollary.

Corollary

A function f from A to B is not $1-1$, if $|A| \geq k+1$ and $|B| = k$.

Examples

1. In any 28 English words, there must be at least three that repeat twice.
2. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day. Assume that classes are not held weekends.
3. Show that among any group of five (not necessarily consecutive) integers, there are two with same remainder when divided by 4.
4. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled om a university to guarantee tat there are at least 100 who come form the same state?
5. There are 28 different time periods during which classes at a university can be schedules. If there are 657 different classes, how many different rooms will be needed?

6.3 Permutations and Combinations

- If we understand permutations and combinations, we can solve many counting problems easily.

1. **A Permutation** is an **ordered** arrangement of distinct objects in a set. An ordered arrangement of r elements of a set is called an **r-permutation**.
2. **A Combination** is an **unordered** selection of distinct objects in a set. An **r-combination** of element of a set is an unordered selection of r elements from the set.

Example1

Let $S = \{a, b, c\}$. List all the permutations and combinations of S. We can think of 3- or 2- permutation and 3- or 2- combination.

- The number of r-permutations and r-combinations of S with $|S| = n$ is denoted by $P(n, r)$ and $C(n, r)$, respectively.

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then the number of r-permutations of set S with $|S| = n$ will be

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1).$$

Example2

1. Let $S = \{a, b, c, d, e, f, g\}$. How many permutations of S end with a?

2. Find the value of the following quantities.

(1) $P(6,3)$ (2) $P(8,8)$ (3) $P(10,9)$

3. Find the number of 5-permutations of a set S with $|S| = 9$

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then the number of r-combinations of set S with $|S| = n$ will be

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

- Use $P(n,r) = C(n,r)P(r,r)$ to prove the previous Theorem.

Corollary

*Let n and r be **nonnegative** integers with $r \leq n$.*
Then $C(n, r) = C(n, n-r)$.

Example3

Find the value of the following quantities.
(1) $C(5,3)$ (2) $C(8,0)$ (3) $C(8,8)$

Example4

1. How many bit strings of length 10 contain
(1) exactly four 1s? (2) at most four 1s?
(3) at least four 1s? (4) an equal number of 0s and 1s?

2. A coin is flipped 10 times where each flip comes up either heads or tales. How many possible outcomes
(1) are there in total? (2) contain exactly two heads?
(3) contain at most three tails?
(4) contain the same number of heads and tails?

3. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?