- When we consider the expansion of powers of binomial expressions, i.e., $(x+y)^n$ for all integers $n \geq 0$, the number of r-combinations is used for all coefficients of the expansion.

Theorem

THE BINOMIAL THEOREM

Let x and y be any variables with any integers $n \geq 0$. Then

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)\, x^{n-j} y^j$$
$$= C(n,0)x^n + C(n,1)x^{n-1}y + \cdots + C(n,n-1)x^n y^{n-1} + C(n,n)y^n.$$

Example1

1. Find the expansion of (1) $(x+y)^4$ and (2) $(x+y)^6$.
2. How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected.
3. What is the coefficient of $x^{101}y^{99}$ in $(2x-3y)^{200}$?

Corollary

Let $n \geq 0$ be integers. Then we have the following identities:

$$\sum_{j=0}^{n} C(n,j) = 2^n,$$

$$\sum_{j=0}^{n} (-1)^j C(n,j) = 0,$$

$$\sum_{k=0}^{n} 2^k C(n,k) = 3^n.$$

- Those identities can be proved easily, using the binomial theorem.

- **Pascal's Identity and Triangle**

Theorem

Pascal's Identity

Let $n > 0$ and $k > 0$ be integers with $n \geq k$. Then

$$C(n+1, k) = C(n, k-1) + C(n, k).$$

- Pascal's identity can be used to recursively define binomial **coefficients**. The identity is the basis for a geometric arrangement of binomial coefficient in a triangle (called **Pascal's triangle**).
- Pascal's identity shows how to arrange a Pascal's triangle.

Example2

What is the row of Pascal's triangle containing the binomial coefficients $C(4, k)$, $0 \leq k \leq 4$?