9. Relations

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Outline of Chapter 9

1. **Relations and Their Properties**
2. $n-$ary relations and their applications
3. **Representing Relations**
4. Closures of Relations
5. **Equivalence relations**

- In our everyday life, we may consider relationships between an employees and their salary, people and their relatives, and so on. In mathematics, we study more clear relationships between numbers, and between a input x and a value of a function $f(x)$, and so on.
- Relationships between elements of sets are represented, using a subset of the Cartesian product of the sets.
- An equivalence relation which is a special type of relations arises throughout mathematics and computer science.

- The mathematical way to expresses a relationship between elements of two sets is to use ordered pairs.

Definition

A binary relation from A to B is a subset of $A \times B$.

- From the definition, a binary relation from A to B is $R = \{(a, b) \mid a \in A \wedge b \in B\} \subseteq A \times B$.
- The notation $a\,R\,b$ means that $(a, b) \in R$ and $a\,(\sim R)\,b$ means that $(a, b) \notin R$.
- When $(a, b) \in R$, a is said to be related to b by R.

Example1

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then a possible relation from A to B is $\{(0, a), (0, b), (1, b), (2, a)\}$. Then $0\,R\,b$ and $1\,(\sim R)\,a$.

- Relations can be represented graphically. We use arrows to represent ordered pairs and use a table to represent relations.

Example2

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2\}$, where $(a, b) \in R$ if and only if
(1) $a = b$ (2) $a \mid b$ (3) $\mathrm{lcm}(a, b) = 2$.

- A function f from A to B can be understood, based on a relation from A to B, since the graph of f is $\{(a, b) \mid b = f(a)\}$. So relations are a generalization of graphs of functions.

Definition

A relation on a set A is a relation from A to A.

- In another word, a relation R on a set A is a subset of $R \subseteq A \times A$.

Example3

Let $A = \{0, 1, 2\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \mid b\}$?

- **Properties of Relations**: let A be a set.

Definitions

1. A relation R on A is called **reflexive** if $(a, a) \in R$ for $\forall a \in A$.
2. A relation R on A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for $\forall a, b \in A$.
3. A relation R on A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$ for $\forall a, b \in A$.
4. A relation R on A is called **transitive** if $(a, c) \in R$ whenever $(a, b) \in R$ and $(b.c) \in R$ for $\forall a, b, c \in A$.

Example4

For each of these relations on the set $\{1,2,3,4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
(1) $\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$
(2) $\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$
(3) $\{(1,1),(2,2),(3,3),(4,4)\}$

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Example5

Determine whether the relation R on $\mathbb{Z}$ is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if (1) $x \neq y$ (2) $xy \geq 1$ (3) $x \equiv y (\text{mod} 7)$ (4) $x \geq y^2$

- **Combining Relations**

Example6

Let $R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$, $R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}$, $R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, $R_4 = \{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$ $R_5 = \{(a, b) \in \mathbb{R}^2 \mid a = b\}$, and $R_6 = \{(a, b) \in \mathbb{R}^2 \mid a \neq b\}$. Then find (1) $R_1 \cup R_2$ (2) $R_4 \cap R_6$ (3) $R_6 - R_3$