9.3 Representing Relations

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- In this section and next section 9.5, we will study only binary relations among all relations.
- We will discuss two alternative methods for representing relations

1. Zero-one matrices are appropriate in computer programs.
2. Pictorial representations called directed graphs are useful for people to get a understanding on the properties of the relations.

- **Representing Relations Using Matrices**
  Suppose that R is a binary relation from $A = \{a_1, a_2, a_3, \cdots, a_m\}$ to $B = \{b_1, b_2, b_3, \cdots, b_n\}$, i.e., a subset of $A \times B$. Then the relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Example1

Represent each of these relations on $\{1, 2, 3\}$ with a matrix.

1. $\{(1,1),(1,2),(1,3)\}$
2. $\{(1,2),(2,1),(2,3),(3,3)\}$
3. $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$
4. $\{(1,3),(3,1)\}$

Example2

List the ordered pairs in the relations on $\{1, 2, 3\}$ for these matrices

$$1.\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad 2.\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- Recall the following properties of relation R on a set A.

1. **Reflexive:** $(a, a) \in R$ for $a \in A$.
2. **Symmetric:** $(a, b) \in R \Rightarrow (b, a) \in R$.
3. **Antisymmetric:** $(a, b) \in R$ **and** $(b, a) \in R \Rightarrow a = b$.

- The **square** matrix $(n \times n)$ of a relation on a set can be used to determine whether the relation has certain properties.

1. **Reflexive:** $m_{ii} = 1$ for $i = 1, 2, \cdots, n$.
2. **Symmetric:** $m_{ij} = m_{ji}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.
3. **Antisymmetric:** $m_{ij} = 1$ with $i \neq j \Rightarrow m_{ij} = 0$. In other words, when $i \neq j$, either $m_{ij} = 0$ or $m_{ji} = 0$.

Example3

Determine whether the relations represented by the matrices in the example 2 are reflexive, symmetric, antisymmetric, transitive.

Example4

How many nozero entries does the matrix correspondeing relation R on $A = \{1, 2, 3, \cdots, 100\}$ consisting of the first 100 positive integers have if R is
1. $\{(a, b) \mid a > b\}$? 2. $\{(a, b) \mid a = b + 1\}$? 3. $\{(a, b) \mid a\,b = 1\}$?

- Suppose that R_1 and R_2 are relations on a set A represented by $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$, respectively. Then

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}, \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

- From the Boolean product, we can define

$$\mathbf{M}_{R_1 \circ R_2} = \mathbf{M}_{R_2} \odot \mathbf{M}_{R_1}, \quad \mathbf{M}_{R^n} = \mathbf{M}_R^n.$$

Example5

Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing (1) R^{-1} (2) $\overline{R}$ (3) R^2

Example6

Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrix representing (1) R^2 (2) R^3

- **Representing Relations Using Digraphs**
  Each element of a set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow.

Definition

A directed graph or digraph consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

- An edge of the form (a,a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.

Example7

List the ordered pairs in the relations represented by the directed graphs. (see Exercises #23, #25, #27 in pp. 597)