9. Equivalence Relations

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- Equivalence relations arise whenever we care only whether an element of a set is in certain class of elements. The "congruence modulo m" relation is an example of equivalence relations.

Definition

A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

- A reason that an equivalence relation is important throughout mathematics and computer science is that it makes sense when two elements of the relation are related.

Definition

Two elements $a, b \in A$ that related by an equivalence relation are called equivalent and denoted by $a \sim b$.

Example1

1. Let R be the relation on the set of rational numbers such that aRb if and only if $a-b$ is an integer. Determine if R is an E. R..
2. Show that the relation is an equivalence relation on $\mathbb{Z}$

$$R = \{(a, b) | a \equiv b (\mathrm{mod}\, 5)\}.$$

Example2

1. Which of these relations on $A = \{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an E. R. that others lack.
(1) $\{(0,0),(1,1),(2,2),(3,3)\}$
(2) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$
(3) $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,2),(3,3)\}$
2. Which of these relations on the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$ are equivalence relations? Do the same thing.
(1) $\{(f, g) \mid f(1) = g(1)\}$.
(2) $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$.

- **Equivalence Classes**

Definition

The equivalence class of A with respect to R is denoted by $[a]_R$:

$$[a]_R = \{s|(a, s) \in R\}.$$

Note that if a relation is obvious, then we write $[a]$ for the equivalence class.

Example3

1. What are the equivalence classes of 1 and 4 for congruence modulo 5?
2. Do the exercise problem #21, #23 on *pp.*616

- **Equivalence Classes and Partitions**

Theorem

Let R be an equivalence on a set A. Then for $a, b \in A$ we have

$$(1)\, aRb \Leftrightarrow (2)\, [a] = [b] \Leftrightarrow (3)\, [a] \cap [b] \neq \emptyset.$$

- A **partition** on a set A is a collection of subsets of A such that 1. $A_i \neq \emptyset$ for $i \in I$ 2. $A_i \cap A_j = \emptyset$ when $i \neq j$. 3. $\cup_{i \in I} A_i = A$.

Theorem

Let R be an equivalence relation on a set A.
Then the equivalence relation classes of R form a partition $\{A_i \mid i \in I\}$ $\Leftrightarrow$ for a given partition $\{A_i \mid i \in I\}$, there is an equivalence relation R that has the set A_i with $i \in I$ as its E.C..

Example4

Which of these collections of subsets are parts. of $\{1, 2, 3, 4, 5, 6\}$?
(1) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ (2) $\{2, 4, 6\}, \{1, 3, 5\}$