1 Taylor Polynomials

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Outline of Chapter 1

- The basic operations of calculators or computers are $+, -, \times, \div$. If we limit ourselves to just those operators, the functions $f(x)$ that we evaluate will be limited to the polynomials ($a_n \neq 0$):

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

  where n is its degree and $a_0, a_1, \cdots, a_n$ are the coefficients.
- Roughly speaking, all elementary (smooth) functions can be approximated by the **Taylor polynomials**.

- Recall the **Taylor series** of the function f centered at $x = a$:

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Then the nth-degree Taylor polynomials of f at $x = a$ denoted by $p_n(x)$ are defined as

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

We know that $p_n(x) \to f(x)$ as $n \to \infty$.

Example1

Let $f(x) = e^x$ and $a = 0$.

1. Find a linear polynomial $p_1(x)$ and quadratic polynomial $p_2(x)$.
2. Find the Taylor polynomial $p_n(x)$.

Example2

1. Let $f(x) = \sin x$ and $a = 0$. Find the Taylor polynomial $p_n(x)$.
2. Let $f(x) = \cos x$ and $a = 0$. Find the Taylor polynomial $p_n(x)$.
3. Let $f(x) = \ln x$ and $a = 1$. Find the Taylor polynomial $p_n(x)$.

- Notations ($\approx$ and $\doteq$) mean "approximately equals".

1. the symbol $\approx$ is used to evaluate functions approximately:

$$2x \approx 5 \text{ or } e^x \approx x + 1.$$

2. the symbol $\doteq$ is generally used with numbers:

$$\pi \doteq 3.141592 \text{ or } \sqrt{192} \doteq 13.8564.$$