1.2 The Error in Taylor's Polynomial

January 27, 2016

When we use the Taylor polynomial to approximate certain functions, we need to consider its accuracy.

Theorem

(Taylor's remainder) Assume that f is a smooth function on an interval $[\alpha, \beta]$ and $a \in [\alpha, \beta]$. Then

$$f(x) - p_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x) \quad \text{for } x \in [\alpha, \beta],$$

where $c_x \in (a, x)$ or $c_x \in (x, a)$. So if $\left|f^{(n+1)}(x)\right| \leq M$ for $x \in [\alpha, \beta]$, the error estimate for the Taylor polynomials is

$$|f(x) - p_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

Example1

Let $f(x) = e^x$ and $a = 0$. Then we have the error estimate at $x = 1$:

$$|e - p_n(1)| \leq \frac{e}{(n+1)!}$$

Moreover for each fixed $x \in (-\infty, \infty)$ $\lim_{n\to\infty} p_n(x) = e^x$, since we can use the Taylor Remainder theorem and

$$\lim_{n\to\infty} M \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for each fixed } x \in (-\infty, \infty).$$

Example2

Use the Taylor remainder theorem to determine how large the number of terms in the Taylor ploynomal has to be chosen to have

$$|e - p_n(1)| \leq 10^{-8}.$$

- For future reference, we have some important Taylor polynomials with the remainder.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{c_x},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \sin c_x,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos c_x,$$

where c_x is in between 0 and x.

- Notations

1. $|f(x) - p_n(x)| \leq M_1$ means that the error caused by the polynomial p_n over the whole domain is bounded by M_1.
2. $\max_{\alpha \leq x \leq \beta} |f(x) - p_n(x)| = M_2$ means that the error caused by the polynomial p_n over the interval $[\alpha, \beta]$ is bounded by M_2. So $|f(x) - p_n(x)| \leq M_1$ is equivalent to $\max_{-\infty < x < \infty} |f(x) - p_n(x)| = M_1$