1.3 Polynomial Evaluation

- Consider polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + \cdots a_n x^n$$

which you need to evaluate for many values of x.

- How do we evaluate the polynomials efficiently?
We use **Nested Multiplication (or Horner's method)** to save the computational time, since fewer operations are required if polynomials are rewritten as N.M(H.M). In order to do so, we group the terms in nested multiplication:

$$\begin{aligned} p(x) &= a_0 + x\left(a_1 + a_2 x + \cdots a_{n-1} x^{n-2} + a_n x^{n-1}\right) \\ &= \vdots \\ &= a_0 + x\left(a_1 + x\left(a_2 + \cdots + x\left(a_{n-1} + x\left(a_n\right)\right)\cdots\right)\right). \end{aligned}$$

- In order to write Scilab code, the polynomial is rewritten in the nested multiplication:

$$p(x) = a_1 + x\left(a_2 + x\left(a_3 + \cdots + x\left(a_n + x\left(a_{n+1}\right)\right)\cdots\right)\right).$$

- Based on the nested multiplication, we can have the following pseudo-code:

$$\begin{array}{l} \textit{real array } (a_i)_{1:n};\ \textit{integer } i, n;\ \textit{real } y, x \\ \qquad \vdots \\ y \leftarrow a_{n+1} \\ \textit{for } i = n : -1 : 1 \\ \qquad y \leftarrow a_i + y * x \\ \textit{end for} \end{array}$$

  The main idea is to start with the inner parentheses and works outward.

- Here is a scilab code for Nested Multiplication:

//Evaluate a poly. $P(x) = a_1 + a_2 x + \cdots + a_{n+1} x^n$ at a value x.
//Input: degree n and an array of $n+1$ coefficients c and a value x.
//Output: value $y = P(x)$.

```
function y=nest_multi(n,c,x)
    y = c(n+1);
    for i=n:-1:1
        y = c(i)+y.*x;
    end
endfunction
```

Example1

Rewrite the following polynomial in nested form.

1. $p(x) = 3x^5 + x^4 - 2x^3 - 5x + 1$
2. $p(x) = 7x^8 - 5x^6 + 3x^3 + 2x^2 - 4$

- Note that we can extend to more general nested form:

$$a_0 + (x - r_0)\,(a_1 + (x - r_1)\,(a_2 + \cdots + (x - r_{n-1})\,(a_n))\cdots),$$

where $r_0, r_1, r_3, \cdots, r_{n-1}$ are called the base points.

Example2

Find the Taylor polynomial $p_5(x)$ for $\ln x$ about $a = 1$. Then rewrite it in nested form.