2 Error and Computer Arithmetic

Dr. Jeongho Ahn

Department of Mathematics & Statistics

ASU

Outline of Chapter 2

1. Floating-Point Numebers
2. **Errors: Definitions, Sources, and Examples**
3. Propagation of Error
4. Summation

- Some Definitions: let x_T denote the true value (exact solution) and x_A denote an approximation of x_T. Then

1. the error in x_A is $\text{error}(x_A) = x_T - x_A$.
2. the relative error in x_A is
$$\text{rel}(x_A) = \frac{\text{error}(x_A)}{x_T} = \frac{x_T - x_A}{x_T} \quad \text{for } x_T \neq 0.$$
3. We will mostly consider the **absolute** error in x_A: $|\text{error}(x_A)|$ and the **absolute** relative error in x_A:$|\text{rel}(x_A)|$.

- For practical reasons, the relative error is more meaningful than the error. We will see it in the next Example1.

Example1

1. If the exact solution $x_T = 2.000$ is and the approximation $x_A = 2.001$, find the absolute error and relative error of x_A.
1. If the exact solution $x_T = 0.001$ is and the approximation $x_A = 0.002$, find the absolute error and relative error of x_A.

- Sources of Error

1. Modelling Errors
2. Blunders and Mistakes
3. Pysical Measurement Errors
4. Machine Errors
5. Mathematical Errors

- Significant digits:
  An approximation x_A has m significant digits w.r.t the true value x_T if $|x_T - x_A|$ is less than or equal to 5 units in the $(m+1)^{\text{st}}$ digit, beginning with the first nonzero digit in x_T.

Example2

1. Let $x_T = 13.27$ and $x_A = 13.24$. Then x_A has 3 significant digits w.r.t. x_T.
2. Let $x_T = e$ and $x_A = 2.71826$. Then x_A has 5 significant digits w.r.t. x_T, since $|x_T - x_A| \doteq 0.00002182$.

- **A loss of significance** occurs when we **subtract** one quantity from the other nearly equal quantity.
- To better understand, we consider the following examples.

Example1

Consider the function

$$f(x) = \sqrt{x^2 + 9} - 3.$$

For $x \approx 0$ we can expect a potential loss of significance in the subtraction. How do we avoid the loss of significance?
Let's calculate $f(0.1)$ on a 3 decimal-digit computer.
1. Without rationalizing f, you will get 0.00, since $\sqrt{9.01} \approx 3.0016$.
2. However, if we rationalize, $f(0.1) = 0.00167 = 1.67 \times 10^{-3}$.

- Here is another example with trigonometric expressions

Example3

Consider the following function

$$g_1(x) = \frac{1 - \cos x}{\sin^2 x} \quad \text{for } x \approx 0$$

In order to avoid loss of significance, we change it into another expression

$$g_2(x) = \frac{1}{1 + \cos x}.$$

- **When we use the quadratic formula**:
  If $4ac \ll b^2$, then one of the roots can be subject to loss of significance.
  For the quadratic equation $ax^2 + bx + c = 0 (a \neq 0)$, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. If $b > 0$ and $4|ac| \ll b^2$,

$$x_1 = -\frac{b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = -\frac{2c}{b + \sqrt{b^2 - 4ac}}$$

2. If $b < 0$ and $4|ac| \ll b^2$,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{2c}{-b + \sqrt{b^2 - 4ac}}$$