3.2 Newton's Method

- This method (called the Newton-Raphson Method) converges much faster than the bisection method does.
- In Newton's method, we assume that $f \in C^1$ and $f'(x_n) \neq 0$ for any $n \geq 0$.
- Newton's method will be obtained, based on the **linear approximation**:

$$x_0 = \text{initial guess}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, 2, 3, \cdots$$

Example1

1. Find the Newton's formula for the equation
$$x^3 = 0.$$
2. Given $x_0 = 1$, find x_1 and x_2 in the Newton iteration.

```
// This code modifies the Newton-Raphson method.
// x0 is an initial point and tol_n is used in stopping the iteration.
function [sol,iter_count,error_bd]=
newton(x0,tol_n,given_fx,deriv_given_fx)
      error_bd=1; iter_count=0;
      while abs (error_bd)>tol_n
          fx=given_fx(x0); //given_fx is given by your test
function
          d_fx=deriv_given_fx(x0);
          // deriv_given _fx is a derivative of test function
          if abs (d_fx)==0 // in order to avoid f' = 0
               disp ('Use the bisection method!');
               return
          end // for if statement
          x1 = x0 - fx/d_fx // Find the next step solution
          error_bd = x1 - x0; x0 = x1;
          iter_count = iter_count + 1; // Count # of iteration.
       end // for while loop
sol = x1;endfunction
```

```
endfunction
/////////////////////////////////////////////////////////
function value1 = test_fun1(x)
value1 = x.^3 - 5;
endfunction
////////////////////////////////////////////////////////////
function value2 = test_fun1_deri(x)
value2 = 3*x.^2;
endfunction
///////////////////////////////////////////////////////////
```

- Convergence Analysis

Definition

Sequence x_n is said to be **quadratically convergent** to a limit x if there is $C > 0$ such that

$$|x_{n+1} - x| \leq C\,|x_n - x|^2 \quad \text{for } n \geq 0.$$

Alternatively, the iteration is quadratically convergent if

$$M = \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|^2} < \infty.$$

Theorem

Suppose that $f \in C^2$, $f(x^*) = 0$, and $f'(x_n) \neq 0$ for all $n \geq 0$. Then Newton's method is locally and quadratically convergent to x^*:

$$|x_{n+1} - x^*| \leq C\,|x_n - x^*|^2 \quad \textit{for } n \geq 0,$$

where $C = |f''(c_n)/(2f'(x_n))|$ and c_n is in between x^* and x_n.

- However, the previous Theorem does not say that Newton's method always converges quadratically.
- We can see it in the next example

Example2

Prove that Newton's method to find root of $f(x) = (x - x^*)^m$ converges linearly, where m is any positive integer.

- The function f is said to have a multiplicity m root of x^* if there is a smallest m such that $f^{(k)}(x^*) = 0$ for $0 \leq k < m$, but $f^{(m)}(x^*) \neq 0$.
- You will see that Newton's method fails in the next examples.

Example3

1. Use Newton's method to find a solution of $-x^4 + 3x^2 + 2 = 0$. Start Newton's method with initial guess $x_0 = 1$
2. Use Newton's method to find a solution of $-x^3 + 2 = 0$. Start Newton's method with initial guess $x_0 = -1$