3.3 Secant Method

- Secant Method replaces the tangent line with an approximation using the secant line.
- Recall the Newton's method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, 2, 3, \cdots$$

- Then approximating $f'(x_n) = (f(x_n) - f(x_{n-1})) / (x_n - x_{n-1})$ provides the Secant method:

$$\begin{aligned} x_0, x_1 &= \text{intial inputs} \\ x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{for } n = 0, 1, 2, 3, \cdots \end{aligned}$$

- Unlike the Newton's method, we need two initial guesses(data) to implement the Secant method.

Example1

Consider the nonlinear $x^3 + 1 = 0$. Apply the Secant Method with two initial guesses $x_0 = 0, x_1 = 1$ to find x_2 and x_3.

```
// This is a code for the secant method
function [appro_sol,it_count]=secant(x0,x1,tol,given_f)
    error_bd = 1; fx0=given_f(x0); it_count = 0;
    while abs(error_bd) > tol
        it_count = it_count + 1;
        fx1=given_f(x1);
        if fx1 - fx0 == 0
            disp('The secant method does not work')
            return
        end // for if
     x2 = x1 - fx1*(x1-x0)/(fx1-fx0);
     error_bd = x2 - x1;
     x0 = x1; x1 = x2; fx0 = fx1;
     end // for while loop
     appro_sol = x2;
endfunction
/////////////////////////////////////////////////
function value = test_func1(x)
value=x.*log(x+2)/3 + sin(x)/2-2; endfunction
```

- Convergence Analysis

Theorem

If $f \in C^3$ *and* $\lim_{n\to\infty} x_n = x^*$ *with* $f'(x_n) \neq 0$*, then*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_{n-1} \cdot e_n} = -\frac{f''(x^*)}{2f'(x^*)},$$

where $e_n = x^* - x_n$.

1. We can also prove that $|x_{n+1} - x^*| \leq C\,|x_n - x^*|^{\alpha}$, where $\alpha = (1 + \sqrt{5})/2$.
2. The Secant Method is called to be **superlinearly convergent**.
3. The Speed of Convergence: Linearly (Bisection method) $<$ Superlinearly (Secant method) $<$ Quadratically (Newton's method)

- **Comparison** of Newton's method and Secant method

1. Newton's method($\alpha = 2$) is faster than the Secant method($\alpha \doteqdot 1.62$). This implies that Newton's method will require fewer iteration to attain a given desired accuracy.
2. Newton's method requires two function evaluation of $f(x_n)$ and $f'(x_n)$ per iteration. So Newton's method is generally more expensive.
3. Decision for choosing which method you want? It depends on you or what types of problems (equations) you have.
4. Note that the bisection method is reliable but slow.