3.4 FPI (Fixed Point Iteration)

- Consider the nonlinear equation

$$x - \cos x = 0.$$

  Then you may use the bisection method or Newton's method(secant method) to find a (approximate) solution of the equation.

- However, the **fixed point equation** $\cos x = x$ which is equivalent to $x - \cos x = 0$ can be considered from a different point of view. From the fixed equation the input x is equal to the output that will be a **fixed point** of $\cos x$. Also the input x is a solution of the equation $x - \cos x = 0$.

Definition

The number x^* is a fixed point of the function g if $g(x^*) = x^*$.

- **FPI** starts with initial guess x_0 and iterates a function g:
  $x_0 =$ initial input
  $x_{i+1} = g(x_i)$ for $i = 1, 2, 3, \cdots$.
- If $g \in C$ and $\lim_{i\to\infty} x_i = x^*$, then $g(x^*) = x^*$.

Example

1. The function $g(x) = \cos x$ has a fixed (approx) point $x^* = 0.73..$
To see it, use the following code:

```
function x=x_final(x_0,k,given_fc) // Output will be an array.
  x(1)=x_0;
  for i=1:k
     x(i+1)=given_fc(x(i));
  end
endfunction
////////////////////////////////////////////
function value = test_func1(x)
   value = cos(x);
endfunction
```

2. Explain why the FPI for $g(x) = \cos x$ converges.

Corollary

Suppose that $g \in C[a, b]$ satisfies the following properties

$$a \leq x \leq b \Rightarrow a \leq g(x) \leq b.$$

Then the equation $x = g(x)$ has at least one solution $x^* \in [a, b]$.

Theorem

(Contraction Mapping Theorem) *Assume that $g \in C^1[a, b]$ and $a \leq g(x) \leq b$ for $a \leq x \leq b$. If $\lambda = \max_{a \leq x \leq b} |g'(x)| < 1$, then*
1. $\exists!$ solution $x^* \in [a, b]$ of the equation $x = g(x)$.
2. For any initial guess $x_0 \in [a, b]$, $\{x_n\}$ converges to x^*.
3.

$$|x^* - x_n| \leq \frac{\lambda^n}{1-\lambda} |x_1 - x_0| \quad \text{for } n \geq 0.$$

4.

$$\lim_{n \to \infty} \frac{|x^* - x_{n+1}|}{|x^* - x_n|} = |g'(x^*)| < 1$$

Definition

It is said that a sequence $\{x_n\}$ converge to x^* with order of convergence $p \geq 1$ if

$$|x^* - x_{n+1}| \leq C \, |x^* - x_n|^p .$$

- If $|g'(x^*)| < 1$, the sequence $\{x_n\}$ converge linearly. Note that the value of $|g'(x^*)|$ is called the linear rate of convergence.
- If $|g'(x^*)| = 1$, there is no conclusion:even if $\{x_n\}$ converges, the speed of convergence is very slow.
- Newton's method is a good example of FPI, since the Newton iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and it can change into $x = g(x)$ with $g(x) = x - f(x)/f'(x)$.