4 Interpolation and Approximation

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Outline of Chapter 4

1. Polynomial Interpolation
2. Error in Polynomial Interpolation
3. Interpolation Using Spline Functions
4. The Best Approximation Problem
5. Chebyshev Polynomials
6. A Near-Minimax Approximation Method
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- **Interpolation:** the process that finds a function satisfying a set of all data.

Definition

The function $y = p(x)$ **interpolates** the data points $(x_1, y_1), \cdots, (x_n, y_n)$ if $y_i = p(x_i)$ for each $1 \leq i \leq n$.

- Note that the function $p(x)$ is called an interpolant which will be any elementary function.
- Note that if $i \neq j$, then $x_i \neq x_j$ for $1 \leq i, j \leq n$, which implies that all points are distinct.
- No matter how many points are given, there is a polynomial $y = p(x)$ that goes through all the points.
- However, in many situations, a polynomial is not satisfactory in practice. Other functions have to be considered, for example, spline functions.
- Interpolation is the reversed of evaluation.

Definition

Lagrange Interpolation

Suppose that n points $(x_1, y_1), \cdots, (x_n, y_n)$ are given. Then the Lagrange interpolating polynomial with degree $n-1$ is

$$p_{n-1}(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots y_n L_n(x),$$

where for $1 \leq k \leq n$ the Lagrange basis function is

$$L_k(x) = \prod_{i \neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

- We notice that $L_k(x_k) = 1$, while $L_k(x_j) = 0$ for $j \neq k$. This can be understood by the Kronecker delta function δ_{kj}:

$$L_k(x_j) = \delta_{kj} = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j. \end{cases}$$

- It is not hard to check that $p_{n-1}(x_k) = y_k$.

Theorem

Let $(x_1, y_1), \cdots (x_n, y_n)$ be n points in the xy-plane. Then there exists a unique polynomial of degree $n-1$ or less than $n-1$ that satisfies $p(x_i) = y_i$ for $i = 1, \cdots, n$.

Example1

1. Find an interpolating for the data points $(0,0), (1,1), (2,3)$.
2. Find the polynomial of degree 2 or less that interpolates the points $(0,1), (1,0), (-1,2)$.

- **Newton's divided differences**

  We list the data points in the following table:

x_1	x_2	$\cdots$	x_n
$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

- Then we define the divided differences:

$$f[x_k] = f(x_k)$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

$$f[x_k, x_{k+1} x_{k+2}, x_{k+3}] = \frac{f[x_{k+1}, x_{k+2}, x_{k+3}] - f[x_k, x_{k+1}, x_{k+2}]}{x_{k+3} - x_k}$$

- The interpolating polynomial by the Newton's divided difference formula is

$$\begin{aligned} p(x) = f[x_1] &+ f[x_1, x_2](x - x_1) \\ &+ f[x_1, x_2, x_3](x - x_1)(x - x_2) \\ &+ f[x_1, x_2, x_3, x_4](x - x_1)(x - x_2)(x - x_3) \\ &+ \cdots \\ &+ f[x_1, \cdots, x_n](x - x_1) \cdots (x - x_{n-1}). \end{aligned}$$

- The recursive formula of the Newton's divided difference provides a convenient table. We can use it to find all coefficients.

Example2

1. Use the divided differences to find the interpolating polynomial passing through the points $(0, -1)$, $(2, 1)$, $(3, 3)$.
2. Adding the fourth data point $(1, 2)$ to the list in the previous problem, find the interpolating polynomial.
3. Use the divided differences to find the interpolating polynomial passing through the points $(0, 2)$, $(1, 1)$, $(2, 0)$, $(3, -1)$. Compare it with the Lagrange interpolation.

- **Newton's divided differences vs the Lagrange interpolation method**

1. The Newton's form is probably the most convenient and efficient. So it is recommended for computing.
2. However, the Lagrange interpolation has several advantage conceptually.