4.2 Error in Polynomial Interpolation

- The interpolating error at each value x is

$$|f(x) - p_n(x)|,$$

where f is the original function and p_n is an interpolating polynomial with $\deg(p_n) = n$.

Theorem

Assume that $p(x)$ is the interpolating polynomial of degree at most n fitting the n distinct points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ and $f \in C^{n+1}[a, b]$. Then the interpolation error is

$$|f(x) - p_n(x)| = \left| \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} \right| \left| f^{(n+1)}(c_x) \right|,$$

where c_x belongs to the open interval $(\min\{x, x_0, x_1, \cdots, x_n\}, \max\{x, x_0, x_1, \cdots, x_n\})$.

Example1:

1. Consider the linear interpolation to $f(x) = e^x$ using two nodes x_0 and x_1 satisfying $0 \leq x_0 < x_1 \leq 1$. Then find the actual error bound of f on the interval $[x_1, x_2]$.
2. If the polynomial $p(x)$ interpolates $f(x) = e^x$ at the points $(-2, -1, 0, 1, 2)$, find the maximum interpolating error at $x = 1/2$ and $x = 3/4$ for $f(x)$.
3. If the polynomial $p(x)$ interpolates $f(x) = 1/(x+1)$ at the points $(0, 1, 2, 3, 4)$, find the maximum interpolating error at $x = 1/2$ and $x = 7/2$ for $f(x)$.

- **Runge Phenomenon**

1. Its specific example is provided by the Runge function:

$$f(x) = \frac{1}{1 + x^2}.$$

2. The interpolating polynomials for the function $f(x)$ wiggle near the ends of the interpolating interval.
3. The Chebyshev interpolation improves the situation.