- **Polynomial Interpolations vs Splines**

1. In the polynomial interpolation, the only one polynomial meets all data points.
2. The idea of splines is to use **piecewise** low degree polynomials to go through all data points.

- The simplest example of a spline is a linear spline which is equivalent to a piecewise linear interpolation.
- Linear splines can be replaced by cubic splines in order to make them smooth curves.
- Note that quadratic splines are rarely used for interpolation. So we mostly consider the **cubic** splines.

- Properties of **Cubic** Splines

1. A cubic spline $S(x)$ through the data points $(x_1, y_1), \cdots, (x_n, y_n)$ is a set of cubic polynomials

$$S_1(x) = y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2]$$
$$S_2(x) = y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3]$$
$$\cdots$$
$$S_{n-1}(x) = y_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \quad \text{on } [x_{n-1}, x_n].$$

2. A cubic spline has three main properties:
(1) $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \cdots, n-1$.
(2) $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \cdots, n-1$.
(3) $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \cdots, n-1$.

- From three properties, we can set up $3n-5$ linear equations. However, we need two more equations to solve for $3n-3$ coefficients. In order to do so, we impose two conditions on two endpoints: (4) $S_1''(x_1) = 0$ and $S_{n-1}''(x_n) = 0$.
- A spline that satisfies the properties (1) – (4) is called a **natural** cubic spline. The two conditions in (4) are called endpoint conditions for a natural conditions. Note that there are many different versions of (4).

Example1

Define the following three cubic splines:

$$S_1(x) = 2 - \frac{13}{8}(x-1) + \frac{5}{8}(x-1)^3 \quad \text{on } [1,2]$$
$$S_2(x) = 1 + \frac{1}{4}(x-2) + \frac{15}{8}(x-2)^2 - \frac{5}{8}(x-2)^3 \quad \text{on } [2,4]$$
$$S_3(x) = 4 + \frac{1}{4}(x-4) - \frac{15}{8}(x-4)^2 + \frac{5}{8}(x-4)^3 \quad \text{on } [4,5]$$

Check that $\{S_1, S_2, S_3\}$ satisfies three properties for the data points $(1,2)$, $(2,1)$, $(4,4)$, and $(5,3)$.

Example2

1. Determine a, b, c, d, e, f, g, h so that $S(x)$ is the natural cubic spline

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d & \text{on } [-1, 0] \\ ex^3 + fx^2 + gx + h & \text{on } [0, 1] \end{cases}$$

with the interpolation conditions (1) $S(-1) = 1$, (2) $S(0) = 2$ and (3) $S(1) = -1$.
2. Find the natural cubic spline through $(0, 3), (1, -2)$, and $(2, 1)$.

- Note that you may use the formulas (4.63-4.65) in page 150 to find $S(x)$. It is not recommendable!

Theorem

If S is the natural cubic spline that interpolates the function $f \in C^2[a,b]$ with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, then

$$\int_a^b [S''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx.$$

Example3

Consider the data

x	0	1	2
y	1	0	3

1. Find the piecewise linear interpolating function for the data.
2. Find the quadratic interpolating polynomial for the data.
3. Find the natural cubic spline that interpolates the data.