4.4 The best approximation problem

- In this section we consider a best possible approximation. For example, how do we improve Taylor approximations (polynomials)?
- Let $f(x)$ be a given function that is continuous on a given interval $[a, b]$. Then we consider approximating it by some polynomial $p(x)$. To measure a possible error in $p(x)$ as an approximation, we introduce the following notation

$$E(p) = \max_{a \le x \le b} |f(x) - p(x)| .$$

  This is called the maximum error or uniform error of approximation of $f(x)$ by $p(x)$ on the interval $[a, b]$. Since the minimax approximation whose degree is lower than the degree of Taylor polynomials can make the small error, we can consider the minimax approximation which provides the 'best' possible approximation of a given $\deg(p) = n$.

- Thus for each $n > 0$, we define

$$\rho_n(f) \equiv \min_{\deg(p)\le n} E(p) = \min_{\deg(p)\le n} \left[\max_{a\le x\le b} |f(x) - p(x)| \right],$$

where the number $\rho_n(f)$ is called the **minimax error**. $\rho_n(f)$ will be the smallest possible uniform error, when we approximate an arbitrary function $f(x)$ by polynomials of degree at most n.

- There is a unique polynomial of degree $\le n$ attaining $\rho_n(f)$ on $[a, b]$. This polynomial is called **minimax polynomial approximation** of order n. If there is a polynomial giving the smallest error, it is denoted by $m_n(x)$, i.e.,

$$E(m_n) = \rho_n(f).$$

Example1

Consider the function $f(x) = e^x$ on $[-1, 1]$ and the linear polynomial approximation to $f(x)$. Then find

$$\max_{-1 \le x \le 1} |e^x - L_1(x)| \text{ and } \max_{-1 \le x \le 1} |e^x - m_1(x)|,$$

where L_1 is the linear Taylor polynomial and $m_1 = 1.1752x + 1.2643$.

- **Accuracy of the Minimax Approximation**
  Assume that a function $f(x)$ is smooth enough on the interval $[a, b]$ and $m_n(x)$ is the minimax approximation of degree n for $f(x)$ on $[a, b]$. Then minimax error satisfies

$$\rho_n(f) \leq \frac{[(b-a)/2]^{n+1}}{(n+1)!2^n} \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|.$$

Example2

Let $f(x) = e^x$ on $[-1, 1]$. Then find the minimax error estimate $n = 1, 2, 10$. Use the Taylor remainder theorem to compare the error estimates.