4.5 Chebyshev Polynomials

- The choice of partitions may have a significant effect on the interpolation error.
- The Chebyshev Interpolation provides a particular optimal way to partition the interval.
- The Chebyshev Interpolation can be used to understand a near-minimax approximation method.

Definition

The *n*th Chebyshev polynomials are defined by

$$T_n(x) = \cos\left(n \cos^{-1} x\right) \quad \text{for } -1 \le x \le 1.$$

Using the substitution $\theta = \cos^{-1} x$, we can simplify the identity;

$$T_n(x) = \cos(n\theta)$$

Example1

Find T_0, T_1, T_2.

- Using the trigonometric identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. we can get the triple recursion relation for the Chebyshev polynomials:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Example2

Using the triple recursion relation, find T_3, T_4, T_5.

- **The Minimum Size Property:**
  For $-1 \le x \le 1$ $|T_n(x)| \le 1$, where $n \ge 0$. We can see easily that

  $$T_n(x) = 2^{n-1} x^n + \text{lower degree terms}, \quad n \ge 1.$$

  Now we modify T_n:

  $$\widetilde{T_n}(x) = \frac{1}{2^{n-1}} T_n(x) = x^n + \text{lower degree terms}.$$

  Then

  $$\left|\widetilde{T_n}(x)\right| \le \frac{1}{2^{n-1}}.$$

  Note that a polynomial whose highest degree term has a coefficient of 1 is called a **monic polynomial.**
- We compare the estimate with $|x^n| \le 1$ for $-1 \le x \le 1$.

- We are led to the following theorem which is useful for improving interpolation scheme.

Theorem

Let $n \geq 1$ be an integer. Consider all possible monic polynomials of degree n. Then the monic polynomial with smallest maximum absolute value on $[-1, 1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$, and its maximum value on $[-1, 1]$ is $1/2^{n-1}$.

- An upper bound for the error made by the Chebyshev Interpolation is usually smaller than for uniformly spaced interpolation.