4.6 A Near-Minimax Approximation Method

- Remember that a uniform spaced set of interpolation nodes on the some interval provides a very poor approximation to the Runge function $f(x) = 1/\left(x^2+1\right)$. In order to improve the situation, we will use the minimax error which is discussed in Section 4.4.
- We consider four nodes x_0, x_1, x_2, x_3 in $[-1, 1]$ and a polynomial $c_3(x)$(near minimax approximation) with $\deg(c_3) \leq 3$ which interpolates $f(x)$. Recall the interpolation error

$$|f(x) - c_3(x)| = \left|\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!}\right| \left|f^{(4)}(c_x)\right|.$$

Then letting $w(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$, we want to choose the nodes x_0, x_1, x_2, x_3 so that

$$\max_{-1\leq x\leq 1} |w(x)|$$

is obtained as small as possible.

- Recall the minimum size property. Then the smallest value of $\max_{-1\le x\le 1}|w(x)|$ is $1/2^3$, since

$$w(x) = \frac{1}{8}T_4(x) = \frac{1}{8}\left(8x^4 - 8x^2 + 1\right).$$

By the Chebyshev polynomials

$$T_4(x) = \cos(4\,\theta) \quad x = \cos\theta.$$

We can find the zeros $4\theta = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, \cdots$, which provides the four approximate nodes

$$x = \pm 0.38283, \quad x = \pm 0.923880.$$

Note that the cos function is even.

Example

Consider the function $f(x) = e^x$ and the interpolating polynomial $c_3(x)$ with the four nodes (by the Chebyshev polynomials) on the interval $[-1, 1]$.
Then we can find

$$\max_{-1 \le x \le 1} |e^x - c_3(x)| = 0.00666$$

and

$$\rho_n(e^x) = m_3(x) = 0.00553.$$

- By the Chebyshev polynomials, the interpolation nodes of the near minimax approximation $c_n(x)$ are obtained by

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \quad \text{for } j = 0, 1, 2, \cdots n,$$

where these $n+1$ nodes are on $[-1, 1]$.