- In the previous section, we considered a near minimax polynomial approximation based on suitable chosen nodes. Another approach is to find an approximation $p(x)$ with a small average error over the interval $[a, b]$. A convenient definition of the average error of the approximation is given by

$$E(p; f) \equiv \sqrt{\frac{1}{b-a} \int_a^b [f(x) - p(x)]^2 \, dx}, \qquad (1)$$

  which is called the root-mean-square-error(**RMSE**) in the approximation of $f(x)$ by $p(x)$.
- Choosing $p(x)$ to minimize $E(p; f)$ is equivalent to minimizing

$$\int_a^b [f(x) - p(x)]^2 \, dx.$$

  The minimizing of (1) is called the least squares approximation problem.

Example1

Let $f(x) = e^x$.
1. Find the polynomial $p(x)$ such that

$$\min_{\deg(p)\le 1} \int_{-1}^{1} [e^x - p(x)]^2 \, dx.$$

2. Find the uniform error of approximation of $f(x)$ by $p(x)$. See the table in the next page.

- **The general case** for the previous example1
  We approximate $f(x)$ on the interval $[a, b]$ and let $n \ge 0$. Seek a polynomial $p_n(x)$ with $\deg(p_n) = n$ such that

$$\min_{\deg(p_n)\le n} g(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n) = \min_{\deg(p_n)\le n} \int_0^1 [f(x) - p_n(x)]^2 \, dx.$$

Errors in Linear Approximations of e^x

Approximation	Maximum Error	RMSE
Taylor poly.	0.718	0.246
Least square poly.	0.439	0.162
Chebyshev poly.	0.372	0.184
Minimax poly.	0.279	0.190

- To minimize $g(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n)$, we invoke the conditions

$$\frac{\partial g}{\partial \alpha_i} = 0 \quad \text{for } i = 0, 1, 2, \cdots, n.$$

By the several algebraic manipulations we can set up the linear system

$$\sum_{j=0}^{n} \frac{\alpha_j}{i+j+1} = \int_0^1 x^i f(x) dx, \quad \text{for } i = 0, 1, \cdots, n.$$

Thus, we can solve the linear system.

- However, we will see that this linear system is **ill-conditioned** and thus is difficult to solve accurately, even for moderately sized values of $n = 5$. As a consequence, this is not a good approach to solving for a minimizer of $g(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n)$.

- **Legendre Polynomials**

1. A class of Legendre polynomials is another better approach to minimizing $E(p; f)$.
2. Legendre polynomials $P_n(x)$ are defined as follows: for $-1 \le x \le 1$

$$P_0(x) = 1,$$
$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right], \quad \text{for } n = 1, 2, 3, \cdots$$

Example2

We can find the following Legendre polynomials

$$P_0(x) = 1 \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \quad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

- For the functions $f(x)$ and $g(x)$ we introduce the following

$$(f,g) = \int_a^b f(x)g(x)dx.$$

- Here are some properties of Legendre polynomials.

1. $\deg(P_n) = n$ and $P_n(1) = 1$
2. The triple recursion relation:

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x), \quad \text{for } n \geq 1.$$

3. Orthogonality and size:

$$(P_i, P_j) = \begin{cases} 0, & \text{if } i \neq j \\ \frac{2}{2j+1} & \text{if } i = j. \end{cases}$$

4. All simple zeros are located in the interval $-1 < x < 1$.
5. Every polynomial $p(x)$ with $\deg(p) \leq n$ can be written by

$$p(x) = \sum_{j=0}^{n} \beta_j P_j(x),$$

where $\beta_0, \beta_1, \cdots, \beta_n$ can be uniquely determined from $p(x)$.

- Using the properties of Legendre polynomials, we can derive the least square approximations of degree n to $f(x)$ on $[-1,1]$:

$$l_n(x) = \sum_{j=0}^{n} \frac{(f, P_j)}{(P_j, P_j)} P_j(x). \tag{2}$$

Example3

Again consider the function $f(x) = e^x$ on $[-1, 1]$. Using (2), we can obtain

$$l_3(x) = 1.175201 + 0.997955x + 0.536722x^2 + 0.176139x^3.$$

Errors in Cubic Approximations of e^x

Approximation	Minimax Error	RMSE
Taylor poly.	0.0516	0.0145
Least square poly.	0.0112	0.0033
Chebyshev poly.	0.0067	0.0038
Minimax poly.	0.0055	0.0039