5.3 Gaussian Numerical Integration(Quadrature)

- In this section, the main goal is to look for numerical integration (quadrature) formulas

$$I(f) = \int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n} w_j f(x_j),$$

where w_j is called the weight and x_j a node point. If the function $f(x)$ is well approximated by polynomials with high degree, $\sum_{j=1}^{n} w_j f(x_j)$ will be quite accurate. Without any restrictions on the node points $\{x_j\}$, we can develop a quite accurate set of numerical integration formulas.

- In order to consider the Gaussian Quadrature, we need the following definition.

Definition

The **degree of precision** of a numerical integration is the greatest integer k for which all degree k or less polynomial are integrated exactly.

Example1

1. The degree of precision of the Trapezoid Rule is 1.
2. Find the degree of precision of the approximation

(1) $\frac{1}{2}[f(0)+f(1)]$ (2) $\frac{1}{3}[f(0)+f(1/2)+f(1)]$ for $\int_0^1 f(x)dx$

1. Case $n = 1$: we want an integration formula

$$I(f) = \int_{-1}^{1} f(x)dx \approx w_1 f(x_1).$$

We can see that the desired formula for $I(f)$ is

$$I(f) = \int_{-1}^{1} f(x)dx \approx 2f(0).$$

2. Case $n = 2$: we want an integration formula

$$I(f) = \int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2).$$

We can also see that the desired formula for $I(f)$ is

$$I(f) = \int_{-1}^{1} f(x)dx \approx f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right).$$

3. Case $n > 2$: we want an integration formula

$$I(f) = \int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) = \sum_{j=1}^{n} w_j f(x_j)$$

- The previous page continues here!
  We need to set up $2n$ equations for the $2n$ unknowns. So we require the quadrature formula to be exact for the cases

$$f(x) = x^i, \quad i = 0, 1, 2, \cdots, 2n-1.$$

Then the following linear system is obtained

$$w_1 x_1^i + w_2 x_2^i + \cdots + w_n x_n^i = \int_{-1}^{1} x^i dx \quad \text{for } i = 0, 1, 2, \cdots, 2n-1.$$

Look at the right side. We have

$$\int_{-1}^{1} x^i dx = \begin{cases} \frac{2}{i+1}, & i = 0, 2, \cdots, 2n-2, \text{(even)} \\ 0 & i = 1, 3, \cdots, 2n-1 \text{(odd)}. \end{cases}$$

The linear system has a unique solution unless the unknowns are reordered. The resulting numerical integration rule from the nonlinear system is called **Gaussian Numerical Integration (quadrature).** In fact, since the nodes and weights are not found by solving this system, they are found by other methods.

# of nodes	nodes x_i	weights w_i
1	0	2
2	$\pm 1/\sqrt{3}$	1
3	$0, \pm\sqrt{3/5}$	$8/9, 5/9$
4	$\pm\sqrt{\frac{3-2\sqrt{6/5}}{7}}, \pm\sqrt{\frac{3+2\sqrt{6/5}}{7}}$	$\frac{18+\sqrt{30}}{36}, \frac{18-\sqrt{30}}{36}$
$\cdots$		

- The Nodes and Weights of Gaussian Quadrature Formulas ($n = 2, 3, 4, \cdots, 8$) are given in the Table 5.7 in page 223.

Example2

Approximate the integrals, using $n = 2$ Gaussian Quadrature. Compare with the exact value and find the error.

1.
$$\int_{-1}^{1} \left(x^3 + 3x^2\right) dx$$

2.
$$\int_{-1}^{1} x^6\, dx$$

3.
$$\int_{-1}^{1} e^x\, dx$$

4.
$$\int_{-1}^{1} \frac{1}{x^2 + 1} dx$$

- **Change of Interval of Integration:** To approximate integrals on a general interval $[a,b]$, we can convert integrals on other finite intervals to integrals over $[-1,1]$:

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{(b-a)t+b+a}{2}\right)\frac{b-a}{2}dt,$$

based on the change of integration variables

$$x = \frac{(b-a)t+b+a}{2}, \quad \text{for } -1 \le t \le 1.$$

Example3

Using Gaussian Quadrature $n=2$, approximate the following integrals:

1.
$$\int_1^4 \ln x\, dx.$$

2.
$$\int_0^2 \frac{2xdx}{x^2+1}.$$

- **Weighted Gaussian Quadrature**

Consider the modified(generalized) integral $I(f)$:

$$I(f) = \int_a^b w(x)f(x)\,dx,$$

where $f(x)$ is a well-behaved (nice) function and $w(x)$ is a possible ill-behaved function. Then Gaussian quadrature can be generalized to handle such integrals $w(x)$ properly. For example,

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx, \quad \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx, \quad \int_0^1 f(x)\log\left(\frac{1}{x}\right)\,dx, \cdots$$

We will seek numerical formulation for the following integral in the next example:

$$I_n(f) = \sum_{j=1}^{n} w_j f(x_j).$$

Example4

Consider the generalized integral

$$I(f) = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx.$$

Here $w(x) = 1/\sqrt{x}$. Then we can find numerical integration formulas.

1. Case $n = 1$: The integration formula has the form

$$I(f) = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx \approx 2f\left(\frac{1}{3}\right).$$

2. Case $n = 2$: The integration formula has the form

$$I(f) = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx \approx 1.304\,f\,(0.116) + 0.742\,f(0.696).$$

3. Case $n > 2$: Have a look at the textbook(p.228).

Example5

It is known that

$$\int_0^1 \frac{\cos(\pi x)}{\sqrt{x}} dx \doteq 0.747966.$$

Then find $I_1(\cos(\pi x))$ and $I_2(\cos(\pi x))$. Find their absolute errors.

- When we consider the weighted Gaussian Quadrature

$$I(f) = \int_a^b w(x)f(x)\,dx \approx \sum_{j=1}^{n} w_j f(x_j) = I_n(f),$$

we require the following conditions on the weight function $w(x)$.

1. $w(x) > 0$ for $a < x < b$.
2. For all integers $n \geq 0$, we have

$$\int_a^b w(x)|x|^n\,dx < \infty.$$