6 Numerical Solutions of O.D.Es

Dr. Jeongho Ahn

Department of Mathematics & Statistics

ASU

Outline of Chapter 6

- There are many ODEs which describe physical situations, i.e, we can see many ODEs in physics or biology or chemistry or engineering or applied sciences or ....
- Many techniques are used in solving DEs. However, there are still many DEs that cannot be solved theoretically. In this chapter, we consider numerical methods for solving DEs.
- The first order differential equations are defined by

$$\frac{dy}{dx} = f(x, y(x)), \quad x \geq x_0.$$

1. For $a, b \in C$, the first order linear DE is defined by $f(x, y(x)) = a(x)y(x) + b(x)$. Then the general solutions can be found in method of integrating factors.
2. Let $a(x) = \lambda$. The general solution of $dy/dx = \lambda y(x) + b(x)$ is

$$y(x) = ce^{\lambda x} + \int_{x_0}^{x} e^{\lambda(x-t)} b(t)\, dt,$$

where c is an arbitrary constant. Then $c = e^{-\lambda x_0} y(x_0)$.

- As you can see in the previous page, we need to impose a condition to obtain a particular solution; $y(x_0) = y_0$.
- In many application problems, the independent variable x is considered as time and thus y_0 is an initial condition.
- The first order DE and the initial condition provide the initial value problem **(IVP)**;

$$\begin{cases} \frac{dy}{dx} = f(x, y(x)) & \text{for } x \geq x_0, \\ y(x_0) = y_0. \end{cases}$$

- **General Solvability Theory**

Theorem

Let $f(x,y), \partial f(x,y)/\partial y \in C((x_0-\alpha, x_0+\alpha)\times(y_0-\varepsilon, y_0+\varepsilon))$. *Then* $\exists!$ *function* $y(x)$ *defined on some interval* $[x_0-\alpha, x_0+\alpha]$ *satisfying the IVP*

$$\begin{cases} \frac{dy}{dx} = f(x, y(x)) & \text{for } x \geq x_0, \\ y(x_0) = y_0. \end{cases}$$

Example1

Is there a unique solution satisfying the following IVP

$$\frac{dy}{dx} = 3x^2 [y(x)]^2 \quad y(0) = 1.$$

If so, find the solution.

- When we deal with differential equations theoretically and numerically, we see the **Lipschitz condition** many times.

Definition

A function $f(t, y)$ satisfies the **Lipschitz condition** in the variable y on $R = [a, b] \times [c, d]$ if $\exists$ constant $L > 0$ such that

$$|f(t, y_2) - f(t, y_1)| \leq L|y_2 - y_1| \quad \text{for } (t, y_1), (t, y_2) \in S.$$

- Note that a function is Lipschitz in $y \Rightarrow$ the function is continuous in y. Is its converse true?

Example2

Prove that $f(t, y) = t\,y + t^2$ is Lipschitz in y for $t \in [0, 1]$.

- **Stability of the IVP** is to consider what happens the solution $y(x)$ if a small change in the data is made.

1. The IVP is called **stable or well-conditioned** if small changes in the data lead to small changes in the solution.
2. The IVP is called **unstable or ill-conditioned** if small changes in the data lead to large changes in the solution.

- In order to determine whether the IVP is stable or unstable, we usually consider its perturbed problem; for $x \in [x_0, b]$

$$\begin{cases} \frac{dy_\varepsilon}{dx} = f\left(x, y_\varepsilon(x)\right), \\ y_\varepsilon\left(x_0\right) = y_0 + \varepsilon. \end{cases}$$

If we show that $|y(x) - y_\varepsilon(x)| \leq \varepsilon M$, the IVP will be stable.

- **First-order linear differential equations:**

$$\frac{dy}{dx} = g(x)\, y(x) + h(x)$$

The solutions will be

$$y(x) = e^{\int g(x)dx} \left(\int e^{-\int g(x)dx} h(x)dx + C \right).$$

Example3

Consider the following IVPs;

$$(1)\quad \begin{cases} \frac{dy}{dx} = -y(x) + 1 \quad \text{for } x \geq 0, \\ y(0) = 1. \end{cases}$$

$$(2)\quad \begin{cases} \frac{dy}{dx} = 10y(x) - 11e^{-x} \quad \text{for } x \geq 0, \\ y(0) = 1. \end{cases}$$

$$(3)\quad \begin{cases} \frac{dy}{dx} = \lambda(y(x) - 1) \quad \text{for } x \geq 0, \\ y(0) = 1. \end{cases}$$

$$(4)\quad \begin{cases} \frac{dy}{dx} = -(y(x))^2 \quad \text{for } x \geq 0, \\ y(0) = 1. \end{cases}$$

Investigate the stability for all IVPs.