6.3 Runge-Kutta (RK) Methods

- In this section, we study a second order RK (Heun's) method to solve the IVP: for $t \in [a, b]$

$$\begin{cases} \frac{dy}{dt} = f(t, y(t)) \\ y(a) = y_a \end{cases}$$

- We use the Taylor series to derive the second order RK. We can begin with it for next step approximations $y(t+h)$, where $h > 0$ is sufficiently small;

$$y(x+h) = y(t) + h\,y'(t) + \frac{h^2}{2!}y''(t) + \frac{h^3}{3!}y'''(t) + \cdots$$

- We take partial derivatives on the first order ODE and use the chain rule repeatedly;

$$y(t+h) = y(t) + \frac{1}{2}(F_1 + F_2),$$

where

$$\begin{cases} F_1 = hf(t, y(t)) \\ F_2 = hf(t+h, y+F_1) \end{cases}.$$

- Let w_n be an approximation at each time step $t = t_n$. Then, we can set up the following iterative formula;

$$w_0 = y(a)$$
$$w_{n+1} = w_n + \frac{1}{2}(F_1 + F_2),$$

where

$$\begin{cases} F_1 = hf(t_n, w_n) \\ F_2 = hf(t_{n+1}, w_n + F_1) \end{cases}.$$