Hermite Interpolation

- For various applications such as higher order PDEs, it is more useful to consider polynomials $p(x)$ that interpolate a **given** function $f(x)$ and have $p'(x)$ interpolate the derivative $f'(x)$.
- We recall that Lagrange interpolation does not include the data of derivatives.
- We consider a simplest and instructive example of Hermite interpolation. Let x_1, x_2 be distinct points. Then a polynomial $p(x)$ whose degree is smallest will satisfy four conditions:

$$p(x_1) = f(x_1),\ p(x_2) = f(x_2),\ p'(x_1) = f'(x_1),\ p'(x_2) = f'(x_2).$$

Thus, $\deg(p) = 3$ at most. We write a cubic polynomial

$$p(x) = a + b(x - x_1) + c(x - x_1)^2 + d(x - x_1)^2 (x - x_2)$$

rather than just writing it to be a standard cubic. This can simplify the work as we can see later. Its derivative is

$$p'(x) = b + 2c(x - x_1) + 2d(x - x_1)(x - x_2) + d(x - x_1)^2.$$

Then, we can determine four unknown quantities.

- When we determine all unknown quantities, some linear systems may be singular. See the following example.

Example1

Find a polynomial p satisfying three conditions:

$$p(0) = 0, \quad p(1) = 1, \quad p'(1/2) = 3.$$

- As seen in the example1, the topic for the general problem associated with such a difficulty is known as **Birkhoff interpolation.**
- In a Hermite problem, let $x_1, x_2, \cdots, x_n$ be nodes. Assume that interpolation conditions are given at x_i with $1 \leq i \leq n$:

$$p^{(j)}(x_i) = c_{ij} \quad \text{for } 0 \leq j \leq k_i - 1. \tag{1}$$

Note that k_i may vary with i. Thus, the total number of conditions on a polynomial p is denoted by m and therefore

$$m = k_1 + \cdots + k_n.$$

We can also see that $m \geq n$ and $\deg(p) = m - 1$ at most.

Theorem

There $\exists!$ polynomial p with $deg(p) = m - 1$ satisfying Hermite interpolation conditions (1).

- **The Extended Newton Divided Difference Method:**
  We can see how the **N. D. D.** method can be extended to Hermite interpolation. Before we start setting up charts, we need to understand how to handle derivative conditions at each node x_i. Let's choose a node, saying $x_i = x$. Since its derivative is the limiting value, we can have

$$\lim_{y \to x} f[x, y] = \lim_{x \to y} \frac{f(y) - f(x)}{y - x} = f'(x).$$

  This justifies first derivatives at the node x

$$f[x, x] = f'(x).$$

  Higher derivatives will be considered by the same way. We will do some examples in the next page.

Example2

Show that (1) $f[x,x,x]=\frac{1}{2!}f''(x)$ (2) $f[x,x,x,x]=\frac{1}{3!}f'''(x)$

- Assuming that there are n occurrences of x, the general form is

$$f[x,x,\cdots,x]=\frac{1}{(n-1)!}f^{(n-1)}(x).$$

- Consider an example to find a polynomial $p_{2n-1}(x)$ such that

$$\begin{aligned} p(x_i)&=f(x_i) \quad \text{for } i=1,2,\cdots,n,\\ p'(x_i)&=f'(x_i) \quad \text{for } i=1,2,\cdots,n. \end{aligned}$$

Based on the N. D. D., we can get the polynomial p

$$\begin{aligned} p(x)=&f[x_1]+f[x_1,x_1](x-x_1)+f[x_1,x_1,x_2](x-x_1)^2\\ &+f[x_1,x_1,x_2,x_2](x-x_1)^2(x-x_2)+\cdots\\ &+f[x_1,x_1,\cdots,x_{n-1},x_{n-1},x_n,x_n](x-x_1)^2(x-x_2)^2\cdots(x-x_n)^2. \end{aligned}$$

We need to understand a pattern for the formula, after setting up tables. If there are higher derivative conditions, we can modify the formula easily. See the next example.

Example3

Use the extended Newton Divided Difference Method.

1. Find a quadratic function p satisfying three conditions:

$$p(0) = 1, \quad p'(0) = 3, \quad p(1) = 2.$$

2. Find a polynomial satisfying five conditions:

$$f(1) = 2, \quad f'(1) = 3, \quad f(2) = 6, \quad f'(2) = 7, \quad f''(2) = 8.$$

3. Find a quartic polynomial that takes the following data:

x	0	1	2
$f(x)$	1	-2	22
$f'(x)$	-3	5	

- **Lagrange Form:**
  Let the nodes be $x_1, x_2, x_3, \cdots, x_n$. We assume that a function value and the first derivative at each node are given. Then, we want to seek a polynomial satisfying data:

$$p(x_i) = c_{i0}, \quad p'(x_i) = c_{i1} \quad \text{for } 1 \le i \le n.$$

By the same way as we did in Lagrange interpolation, we can write

$$p(x) = \sum_{k=1}^{n} c_{k0} A_k(x) + \sum_{k=1}^{n} c_{k1} B_k(x),$$

where A_k and B_k satisfy the following property

$$\begin{cases} A_k(x_j) = \delta_{kj}, \\ A'_k(x_j) = 0, \end{cases} \qquad \begin{cases} B_k(x_j) = 0, \\ B'_k(x_j) = \delta_{kj}. \end{cases}$$

Then we can employ the Lagrange basis function L_k with $1 \le k \le n$ to prove that

$$\begin{cases} A_k(x) = \left(1 - 2(x - x_k) L'_k(x_k)\right) [L_k(x)]^2, \\ B_k(x) = (x - x_k) [L_k(x)]^2. \end{cases}$$

- Since the degree of each basis function L_k is $n-1$, the degree of A_k and B_k is $2n-1$. Thus, $\deg(p) = 2n-1$ at most. Equivalently, we need $2n$ conditions on the polynomials p.
- For example, consider p satisfying all data

$$p(x_1) = f(x_1), \quad p(x_2) = f(x_2), \quad p'(x_1) = f'(x_1), \quad p'(x_2) = f'(x_2).$$

Then, the Lagrange form of the polynomial p is written by

$$p(x) = f(x_1)A_1(x) + f(x_2)A_2(x) + f'(x_1)B_1(x) + f'(x_2)A_2(x),$$

where A_1, A_2, B_1, B_2 will be determined.

Example4

Find a polynomial p such that $\deg(p) \leq 3$ and

$$p(0) = 1, \quad p'(0) = 0, \quad p(1) = 2, \quad p'(1) = -1.$$

(1) Use the Lagrange form
(2) Use the extended N. D. D to check your answer.