10.2 Heat Equation on a unbounded domain

- We consider the homogeneous heat equation on **unbounded** intervals:

  \[ u_t = k u_{xx} \quad \text{in} \quad (0, \infty) \times (-\infty, \infty), \]  
  \[ u(t, -\infty) = u(t, \infty) = 0, \quad \text{on} \quad (0, \infty), \]  
  \[ u(0, x) = u^0(x) \quad \text{in} \quad (-\infty, \infty). \]  

- Without using separation of variables, we can check easily that the solution of (1) is

  \[ u(t, x) = e^{-i\omega x} e^{-k\omega^2 t}. \]

Why do we need the complex function \( e^{-i\omega x} \) in the solution? What about \( e^{i\omega x} \) or even real function \( e^{\omega x} \)? We will see the reason later on.
Separation of variables
We use separation of variables to set up $u(t, x) = \phi(t)\psi(x)$. As we did in the finite domain $[0, L]$, we can set up two eigenvalue problems

$$\phi'(t) - \lambda k \phi(t) = 0,$$
$$\psi''(x) + \lambda \psi(x) = 0.$$  

At this stage, we put the boundary condition $|\psi(\pm\infty)| < \infty$, which is rather strange. However, our eventual solution will satisfy the original boundary conditions (2) after superposition.

Remember that our choice for $\lambda$ is $\lambda < 0$. Unlike the discrete eigenvalues $\lambda_n$ for the heat equation in the finite interval, we will use the continuous eigenvalues (spectrum). Why? we will see it soon.
Superposition principle: instead of summing over all $\lambda_n < 0$, we integrate over the continuous spectrum:

$$u(t, x) = \int_{-\infty}^{0} \left( A_1(-\lambda) \cos \left( \sqrt{-\lambda} x \right) + B_1(-\lambda) \sin \left( \sqrt{-\lambda} x \right) \right) e^{\lambda k t} d\lambda$$

For a better form, let $\lambda = -\omega^2$ so that

$$u(t, x) = \int_0^{\infty} A_2(\omega) \cos(\omega x) e^{-k\omega^2 t} + B_2(\omega) \sin(\omega x) e^{-k\omega^2 t} d\omega. \quad (4)$$

This is analogous to the periodic extension of piecewise smooth functions for the finite intervals:

$$u(t, x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right] e^{-k(n\pi/L)^2 t}.$$

Using the initial condition (3), it follows from (4) that

$$u(0, x) = u^0(x) = \int_0^{\infty} A_2(\omega) \cos(\omega x) + B_2(\omega) \sin(\omega x) d\omega.$$

How to determine the coefficient function $A_2$ and $B_2$?
The Complex form of solutions

As we can see the solution in (4), we can write it to be a complex form, using $e^{-i\omega x}$ or $e^{i\omega x}$. We may consider only $e^{-i\omega x}$, alternately. By the generalized principle of superposition (integration), the solution of heat equations becomes

$$u(t, x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k \omega^2 t} d\omega.$$ 

which is equivalent to (4). In this form, we use the initial data (3) to set up the following

$$u(0, x) = u^0(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega.$$ 

How to determine the coefficient functions $c(\omega)$?