

Second-Order Systems: Vibrating Cantilever Beams

Second Order Systems

A second-order system is governed by the second-order ordinary differential equation

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = F(t) \quad (10)$$

where $y(t)$ is the response (output) of the system to an applied force $F(t)$ (input), a_0 , a_1 , and a_2 are system parameters. By rewriting Eq. (10) as

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = k\omega_n^2 F(t) \quad (11)$$

the system parameters become

$$\omega_n = \sqrt{\frac{a_0}{a_2}} \quad \text{undamped natural frequency} \quad (12)$$

$$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}} \quad \text{damping ratio} \quad (13)$$

$$k = \frac{1}{a_0} \quad \text{gain} \quad (14)$$

Consider the homogenous equation in which the applied force $F(t) = 0$

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = 0 \quad (15)$$

that has solutions of the form

$$y_h(t) = \mathbf{C}e^{\lambda t}. \quad (16)$$

Upon substitution into Eq. (15), we obtain the characteristic equation

$$\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0. \quad (17)$$

In general there are two roots, λ_1 and λ_2 :

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \quad (18)$$

The behavior of the system depends on whether these roots are real or complex, that is, whether the **damping ratio**, ζ , is less than, equal to, or greater than 1.

Under-damped Case: $0 < \zeta < 1$

The general solution to the homogenous equation when the damping ratio is less than 1 is constructed from a linear combination of the two solutions with the two complex roots,

$$y_h(t) = C_1 e^{(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2})t} + C_2 e^{(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2})t} \quad (19)$$

where C_1 and C_2 are determined from the initial conditions. The above equation may be expressed in either of the two forms

$$y_h(t) = C e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad (20)$$

$$y_h(t) = e^{-\zeta\omega_n t} (C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)) \quad (21)$$

These equations describe an exponentially decaying sinusoidal response with frequency

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \quad \text{damped natural frequency} \quad (22)$$

and phase shift ϕ . Thus, for zero damping $\omega_d = \omega_n$ and the response is oscillatory, as shown in Fig. 2.

Critically Damped Case: $\zeta=1$

When the damping ratio equals 1, there is only one real root of characteristic equation (17): $\lambda_1 = \lambda_2 = -\zeta\omega_n = -\omega_n$. For repeated roots the second solution is given by

$$y_h = C \omega_n t e^{-\omega_n t}. \quad (23)$$

The general solution to the homogenous equation is formed by adding the two independent solutions to obtain

$$y_h(t) = e^{-\omega_n t} (C_1 + C_2 \omega_n t) \quad (24)$$

where C_1 and C_2 are determined from the initial conditions. Thus, for a critically damped system, the response is not oscillatory; the response approaches equilibrium as quickly as possible as shown in Fig. 2.

Over-damped Case: $\zeta > 1$

When the damping ratio is greater than one, the roots of the characteristic equation are real. The general solution to the homogenous equation is

$$y_h = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}. \quad (25)$$

Thus, for large damping, the response is heavily damped, as shown in Fig. 2.

Figure 2 shows the response for various values of the damping ratio, including under-damped (Eq. 21), critically damped (Eq. 24), and over-damped (Eq. 25) systems.

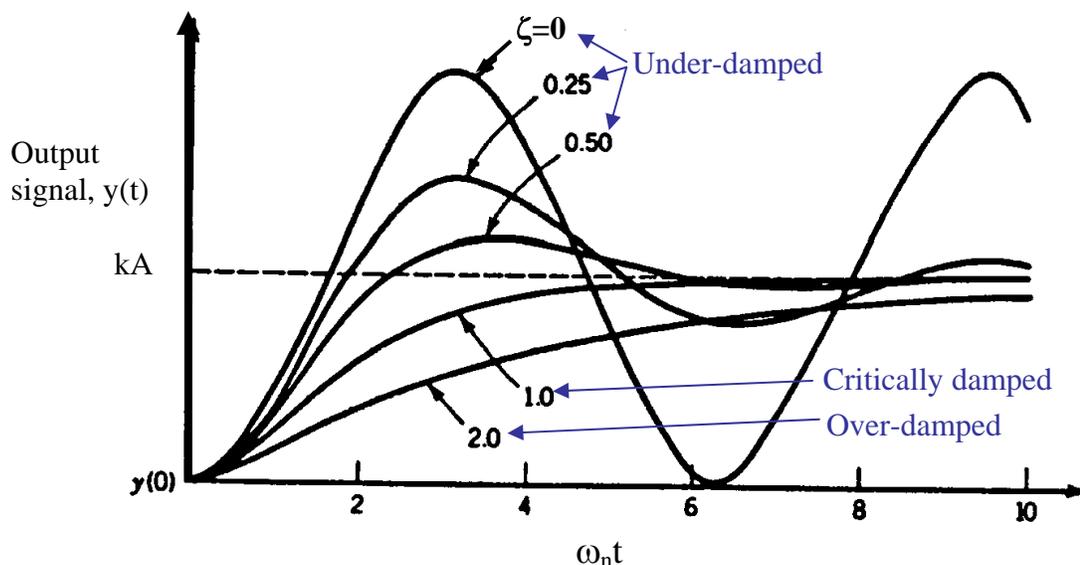


Figure 2: Response of a second-order system to a step input for different damping ratios.

Response to a Step Input

Consider a force instantaneously applied to a second-order system at time $t = 0$:

$$F(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases} \quad (26)$$

The solution to Eq. (15) with this forcing function may be constructed by adding the solution to the homogenous equation to the particular solution, $y_p = kA$ and applying appropriate initial conditions on $y(0)$ and $\dot{y}(0)$.

For the **underdamped** case, this will result in a solution of the form

$$y = C e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad (27)$$

that is an exponentially decaying sine wave. This form of the solution allows determination of the damping ratio from the system's response to a step input by examination of the relative amplitude as the wave decays. Let y_n be the amplitude of the n th peak, which occurs at time t_n as in Figure 3.

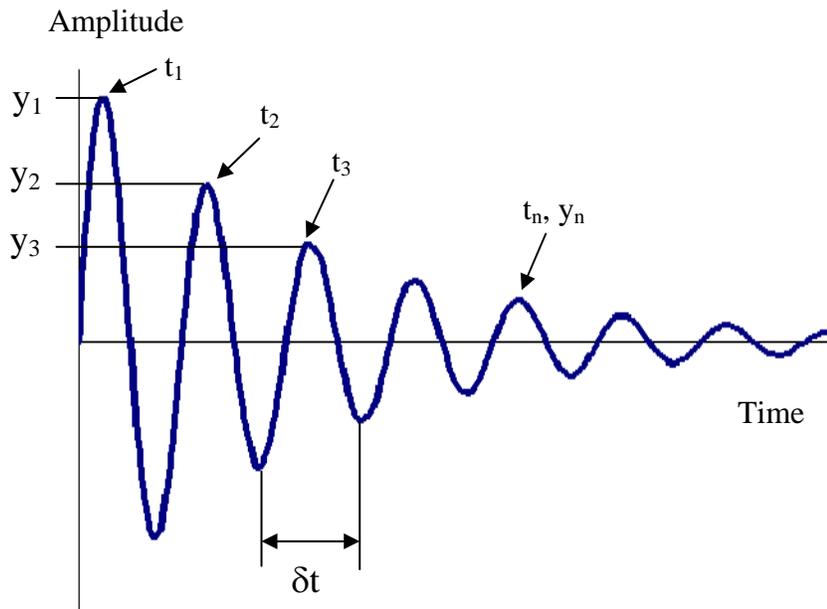


Figure 3: An under damped second order system response showing the amplitude of the peaks for use in calculating damping ratio.

Next we define the *logarithmic decrement*, δ , using the ratio of two peak heights separated by m successive cycles of period T

$$\delta = \frac{1}{m} \ln \frac{y(t_n)}{y(t_{n+mT})} \quad (28)$$

At the peaks, the sine portion of the response equals 1, so by substitution of Equation (27) into (28) we have

$$\delta = \zeta \omega_n T \quad (29)$$

T is the period of the damped free vibration, which can be related to the period of the undamped free vibration (Eq. 22). Thus

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad (30)$$

Once δ has been calculated from the peaks (Eq. 28), ζ can be calculated from δ .

References

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Transient Response of a Cantilever Beam

Design engineers must often consider mechanical vibrations in structures. If a structure is excited at its resonance frequency and the damping is low, excessive vibrations in the structure can lead to catastrophic failure. The natural frequencies of the structure, the damping in the structure, and the frequencies of likely excitations in service must be known in order to assure the reliability of the structure.

In this part of the experiment we will study the dynamic behavior of a simple structure that may be modeled as a second-order system. A cantilever beam is shown in Fig. 4, neglecting rotary inertia and shear effects. We start with a full model, using a fourth-order partial differential equation, with both x and t as independent variables, and then reduce it to a second-order ordinary differential equation in t . The fourth-order differential equation for the deflection $y(x,t)$ is given by

$$EI \frac{\partial^4}{\partial x^4} y(x,t) + m \frac{\partial^2}{\partial t^2} y(x,t) = q(x,t) \quad (31)$$

where E is Young's modulus of the beam material, I is the area moment of inertia of the cross-section, m is the mass per unit length, and $q(x,t)$ is the force per unit length acting in the y direction.

Consider the free vibration of the beam, $q(x,t) = 0$. This partial differential equation may be solved by the method of separation of variables,

$$y(x,t) = Y(x)W(t) \quad (32)$$

which leads to the two ordinary differential equations

$$\frac{d^2W}{dt^2} + \omega_n^2 W = 0 \quad (33)$$

$$\frac{d^4Y}{dx^4} - \beta^4 Y = 0 \quad (34)$$

where

$$\beta^4 = \frac{\omega_n^2 m}{EI}. \quad (35)$$

By comparing Eq. (33) to Eq. (11), we see that the deflection is second-order in time with an undamped natural frequency ω_n and there is no damping included in the model.

The natural deflection shapes (*modes*) of the beam are found by solving Eq. (34) subject to the appropriate boundary conditions. The general solution is

$$Y(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \quad (36)$$

where the constants C_1 , C_2 , C_3 , C_4 , and β are determined by imposing the boundary conditions for a cantilever beam,

$$Y(0) = 0 \quad \text{no deflection at the clamped end,} \quad (37)$$

$$\left. \frac{d}{dx} Y(x) \right|_{x=0} = 0 \quad \text{slope of the beam is zero at the clamped end,} \quad (38)$$

$$M(L) = EI \left. \frac{d^2}{dx^2} Y(x) \right|_{x=L} = 0 \quad \text{no moment at the free end,} \quad (39)$$

$$V(L) = EI \left. \frac{d^3}{dx^3} Y(x) \right|_{x=L} = 0 \quad \text{no shear force at the free end,} \quad (40)$$

which leads to the condition

$$\cos \beta L \cosh \beta L = -1. \quad (41)$$

This transcendental equation must be solved numerically to determine allowable values of β . There are an infinite number of solutions corresponding to the possible modes of vibration, the first three of which are shown in Fig. 5. Solving Eq. (34) for ω_n gives the undamped natural frequencies of the cantilever beam, of which the first two modes are:

$$\omega_{n1} = (1.875)^2 \sqrt{\frac{EI}{mL^4}} \quad \text{and} \quad \omega_{n2} = (4.694)^2 \sqrt{\frac{EI}{mL^4}} \quad (42)$$

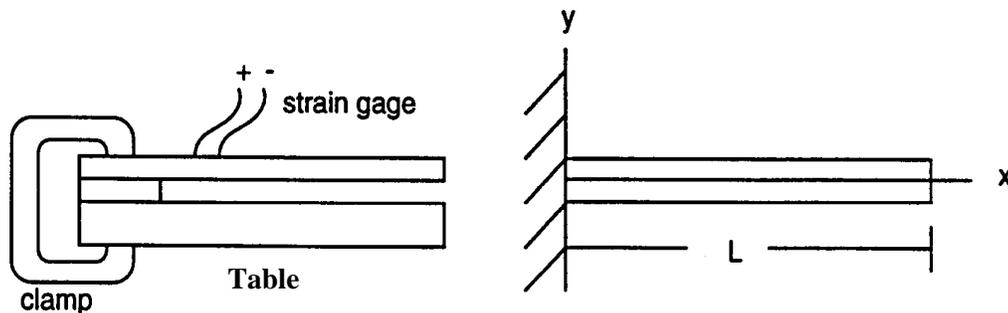


Figure 4: Cantilever beam apparatus and model.

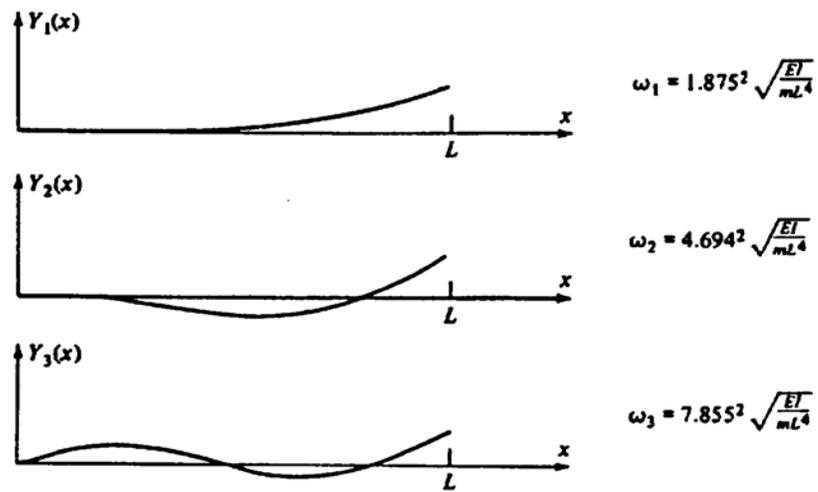


Figure 5: First three modes of vibration of a cantilever beam.