Second Order Measurement System Behaviour

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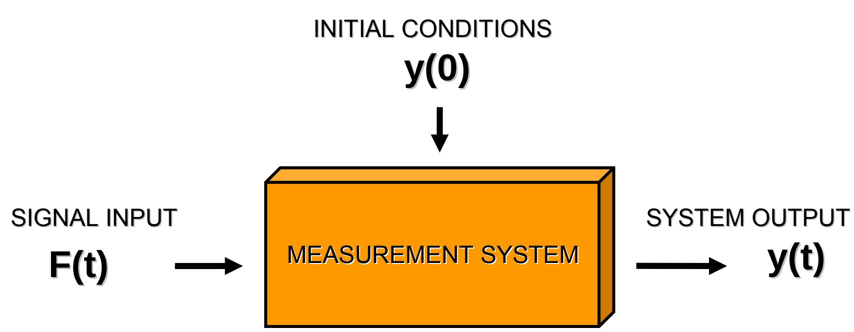
GENERAL MEASUREMENT SYSTEM MODEL



Understanding the theoretical response of our measurement system is essential for

- Specification of appropriate transducers
- Understanding our measurement results

However: Exact response is always determined and confirmed through calibration

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GENERAL MEASUREMENT SYSTEM MODEL

• Static measurement: measured physical quantity is NOT changing with time

• Dynamic measurement: It is ! Consider a vibrating beam – the displacement is constantly changing with time. Or fluid flow under turbulent conditions.

• We will discuss some important parameters and characteristics applicable to a measurement system under dynamic conditions

• A general form of describing a system: through an n^{th} order non-homogeneous differential equation

• The order of the system is designated by the order of the differential equation

• Right-hand side: Forcing function *F(t)* imposed on the system

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WE WILL CONSIDER THREE GENERAL SYSTEM MODELS

1. ZERO ORDER SYSTEM
    - General Model
    - Response to unit step input
2. FIRST ORDER SYSTEM
    - General Model (recap from temperature unit)
    - Response to unit step input
3. SECOND ORDER SYSTEM
    - General Model
    - Response to unit step input
    - Response to vibration or sinusoidal inputs

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Zero Order System

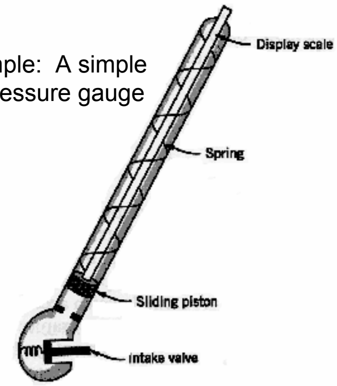
Defn: A system whose behaviour is independent of the time-dependent characteristics of storage or inertia.

Figliola and Beasley, Theory and Design for Mechanical Measurements

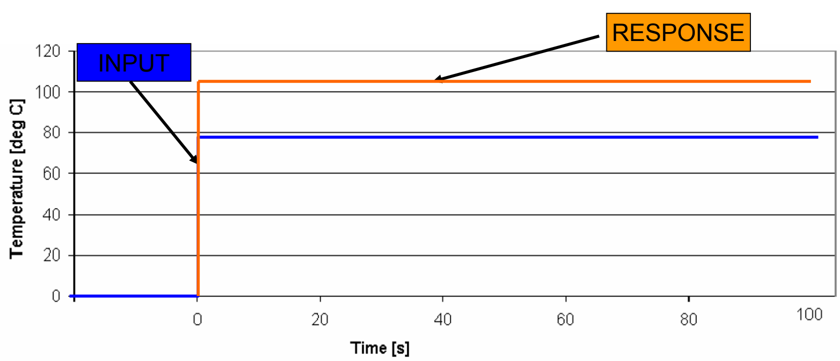
Characteristics of Zero Order System

- Suitable for static signals only
- System has negligible inertia
- System has negligible damping
- Output = constant x input

Example: A simple pressure gauge



Response of Zero Order System



RESPONSE:

- Instantaneous response
- Output tracks input exactly
- Characterised by a zeroth order differential equation

Zero Order Response

$$a_0 x = F(t)$$

Zero Order System

Zero Order Response

$$a_0 x = F(t)$$

Zeroth Order Response described by the equation above:

$$x(t) = (1/a_0)F(t)$$

- System variable *x(t)* will track the input force *F(t)* by a constant (a_0)
- The constant $1/a_0$ is called the static sensitivity of the system
- e.g. if a constant force were applied to a beam, the static deflection will be equal to F/a_0

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First Order System

A system characterised as having time-dependent storage or dissipative ability but having no inertia.

Figliola and Beasley, Theory and Design for Mechanical Measurements

For example: A temperature sensor exhibits a first order response

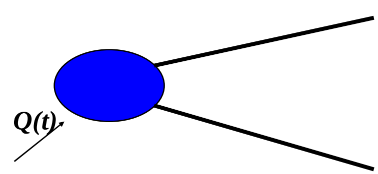
- Response is dictated by heat transfer to transducer

General Response Characterised by first order differential equation:

First Order Response

$$a_1 \dot{y}(t) + a_0 y(t) = F(t)$$

Where a_1 and a_0 are constants



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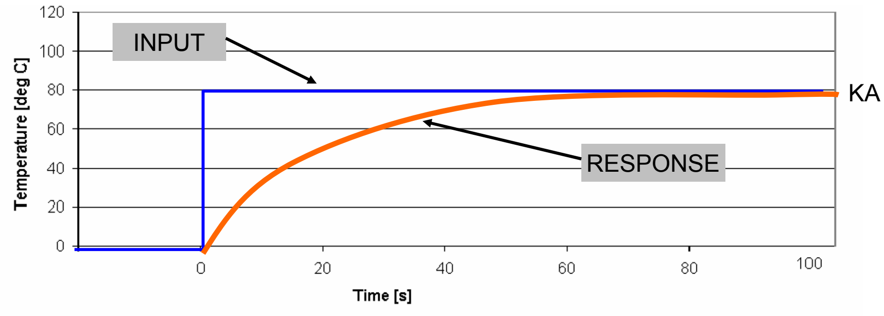
First Order Response to Step Input



Response of First Order System to Step Input:

$$\underbrace{y(t) = KA}_{\text{Steady state component}} + \underbrace{(y_0 - KA)e^{\frac{-t}{\tau}}}_{\text{Transient component}}$$

← Solution to first order differential equation for step function input

Where time constant $\tau = a_1/a_o$

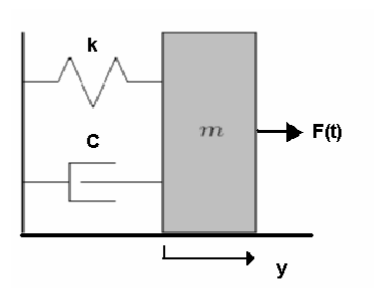
Transducers with Mass, Inertia & Damping

Second Order System

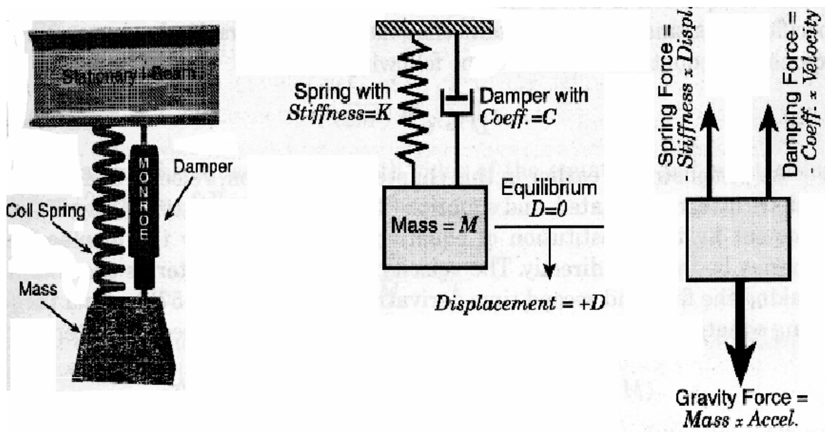
A system whose behaviour includes time-dependent inertia

Figliola and Besley, Theory and Design for Mechanical Measurements

- Pressure and Acceleration transducers
- Motion of a transducer element is coupled to parameter under observation
- Using the Single Degree-of-Freedom System Model

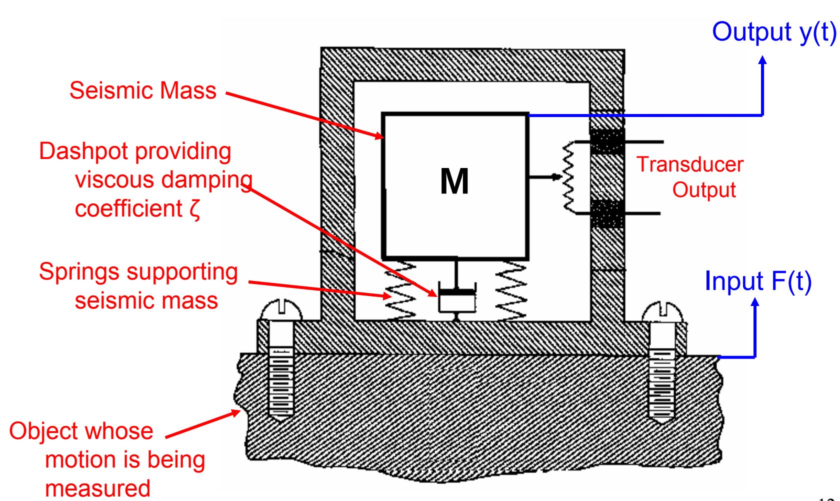


The Mass-Spring-Dashpot Model

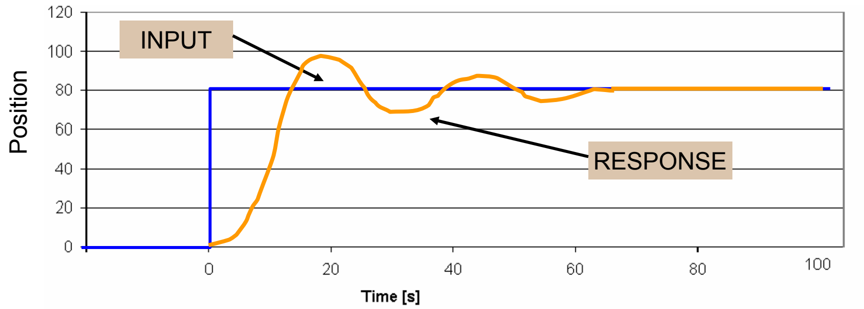


Example of Second Order Transducer

Seismic Transducer



Response of a Second Order System



The Importance of Damping

- Damping governs the ability of our instrument to follow a changing measurement
- Too much damping slows response
- Too little results in excess oscillations from small disturbances and excessive **ringing**
- **A balance must be struck between ringing and settling time**

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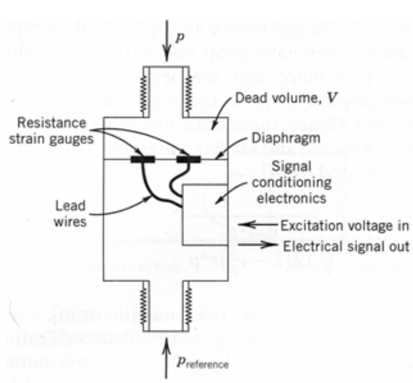
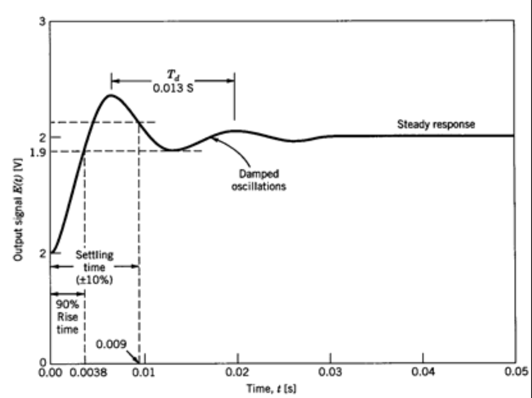
Example of Second Order Transducer

Pressure Transducer





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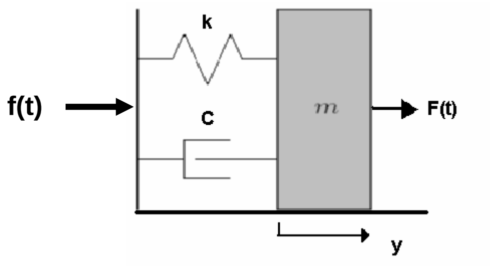
The equation of motion comes from the force balance equation



$$M\ddot{y}(t)+C\dot{y}(t)+Ky(t)=F(t)$$

In terms of the nomenclature in our text book:

$$a_2\ddot{y}+a_1\dot{y}+a_0y=F(t)$$

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Second Order System

- Possesses inertia and contains a second deviative term, such as accelerometers, diaphragm pressure transducers, and acoustic microphones.

$$a_2\ddot{y}+a_1\dot{y}+a_0y=F(t) \quad \text{or}$$

$$(1/(w_n^2))\ddot{y}+(2\zeta\dot{y})/w_n+y=KF(t)$$

where

$w_n = \sqrt{a_0/a_2}$ = natural frequency

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Second Order System

- $\zeta = a_1/(2(a_0 a_2)^{1/2})$ Zeta= damping ratio
- The damping ratio is a measure of system damping, a property of a system that enables it to dissipate energy internally.

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Second Order System

Homogeneous solution

- Quadratic equations have two roots

$$1/(w_n^2)\lambda^2 + (2\zeta/w_n)\lambda + 1 = 0 \quad >> \lambda_1, \lambda_2$$

$$\lambda_1, \lambda_2 = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

- Homogeneous solution gives us the transient response
- Finds the roots of the characteristic equation

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Second Order System

- The three forms of homogeneous solution depend on the value of damping
- $0<\zeta<1$ (underdamped) – oscillatory response
  $$y_h(t) = Ce^{-\zeta w_n t}\sin(w_n\sqrt{1-\zeta^2}t + \theta$$
- $\zeta=1$ (critically damped- asymptotically approaches SS)
  $$y_h(t) = C_1e^{\lambda_1 t} + C_2te^{\lambda_2 t}$$
- $\zeta>1$ (overdamped)
  $$y_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$$

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Properties of 2nd Order Systems

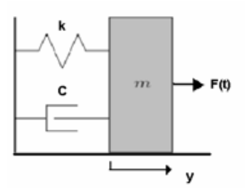
Natural Frequency of System $[\omega_n]$

The frequency of the free oscillations of a system

Figliola and Beasley, Theory and Design for Mechanical Measurements

We define the natural frequency for a system as:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \omega_n = \sqrt{\frac{a_0}{a_2}}$$



We define the damping ratio for a system as:

$$\zeta = \frac{a_1}{2\sqrt{a_0 \cdot a_2}}$$

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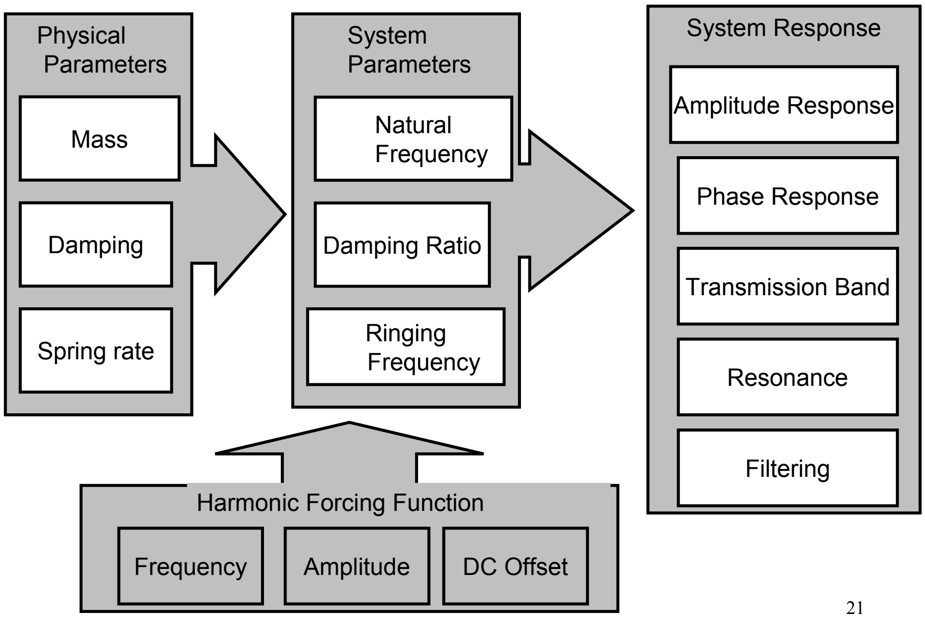
Second Order Response



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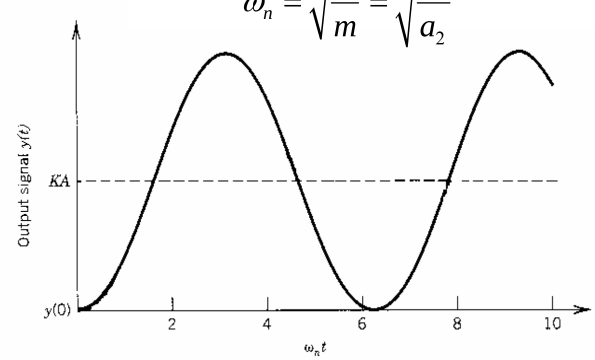
2nd Order Response to Step Input

Case 1: Undamped

Undamped Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{a_0}{a_2}}$$

$$\zeta = 0$$



Response to step input:

Initial disturbance produces harmonic motion which continues indefinitely

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2nd Order Response to Step Input

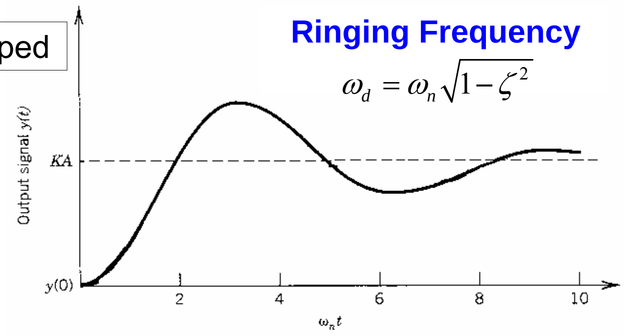
Case 2: Underdamped

Ringing Frequency

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$0 < \zeta < 1$$

$$\zeta = 0.25$$



Response : Displacement response overshoots the steady-state value initially and then eventually decays to steady state value

$$y(t) = KA - KAe^{-\zeta\omega_n t} \cdot \left[\frac{\zeta}{(1-\zeta^2)^{1/2}} \sin(\omega_n \sqrt{1-\zeta^2} \cdot t) + \cos(\omega_n \sqrt{1-\zeta^2} \cdot t) \right]$$

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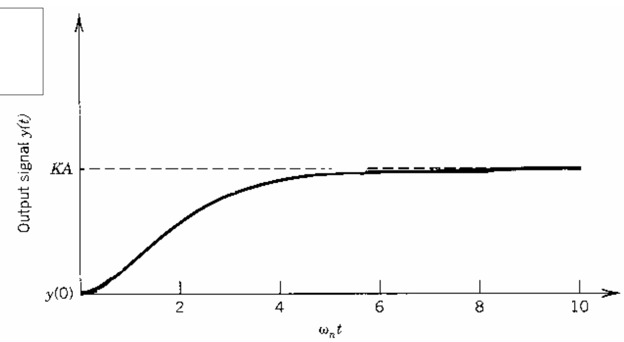
2nd Order Response to Step Input

Case 3: Critically Damped

$$\zeta = 1$$



Response: An exponential rise occurs to approach the steady state value without overshooting it

$$y(t) = KA - KA(1+\omega_n t)e^{-\omega_n t}$$

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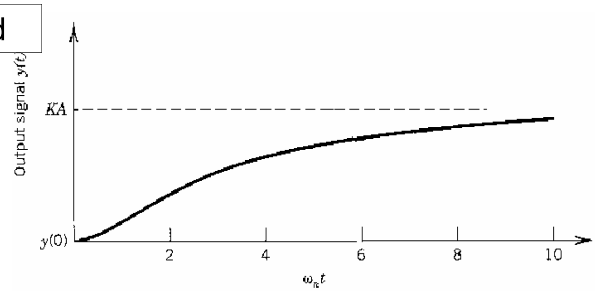
2nd Order Response to Step Input

Case 4: Overdamped

$$\zeta > 1$$



Response: System approaches the steady state value at a slow rate with no overshoot

$$y(t) = KA - KA \cdot \left[\frac{\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + \frac{\zeta - \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \right]$$

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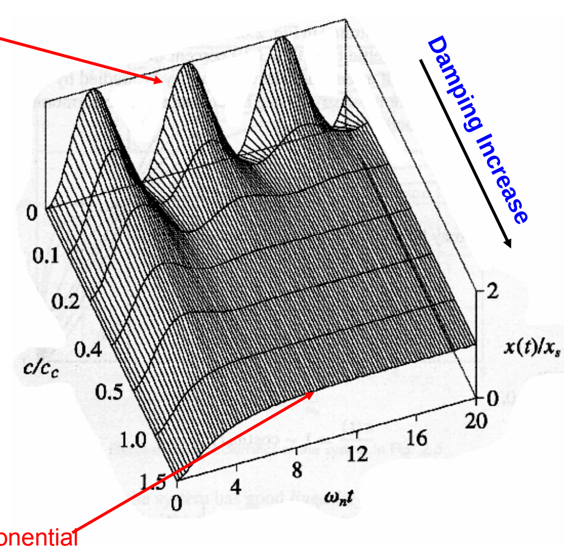
2nd Order Response to Step

We can see here the contrast between harmonic behaviour and exponential behaviour for the ranges of damping



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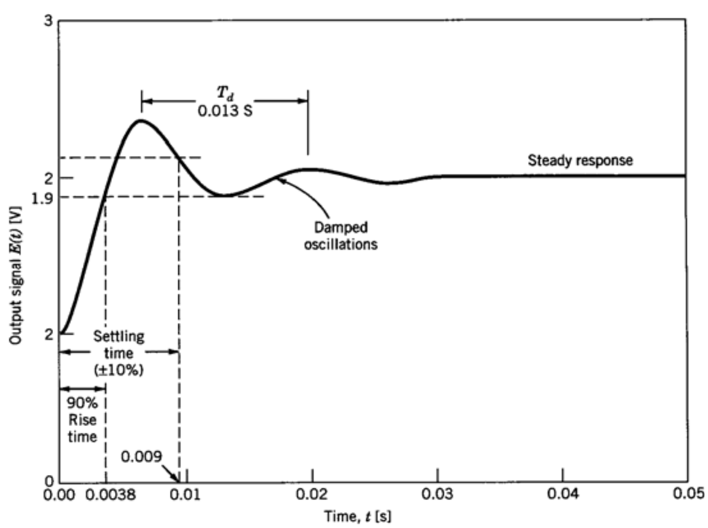
2nd Order Response

1. Duration of transient response controlled by $\zeta\omega_n$. For $\zeta>1$, the response $y_\infty \rightarrow KA$ at $t \rightarrow \infty$, but for larger $\zeta\omega_n$ the response is faster.
2. Time required to reach 90% of step input $Au(t)=KA-y_0$ is called rise time. Rise time is decreased by decreasing the damping ratio ζ.
3. Time to reach ±10% of steady state is called settling time for oscillatory systems.

Note: *a faster rise may not necessarily reach a steady state faster if the oscillations are large.*

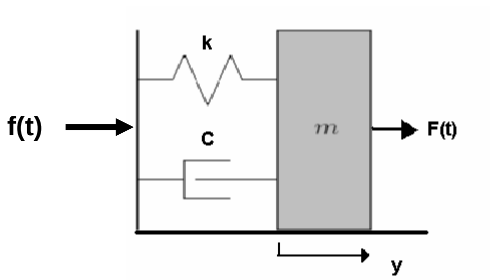
2nd Order Response



Second Order Measurement System Behaviour

Response to Harmonic Excitation

The equation of motion comes from the force balance equation



$$M\ddot{y}(t)+C\dot{y}(t)+Ky(t)=F(t)$$

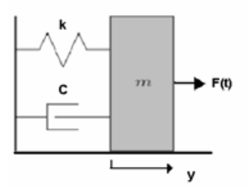
In terms of the nomenclature in our text book:

$$a_2\ddot{y}+a_1\dot{y}+a_0y=F(t)$$

Natural Frequency of System $[\omega_n]$

Defn: The frequency of the free oscillations of a system.

Figliola and Besley, Theory and Design for Mechanical Measurements



We define the natural frequency for a system as:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \omega_n = \sqrt{\frac{a_0}{a_2}}$$

EXPERIMENT: To find natural frequency of an undamped spring mass system, allowed to oscillate:

1. Set the system to vibrate naturally
2. $$\omega_n = \frac{cycles}{time}$$

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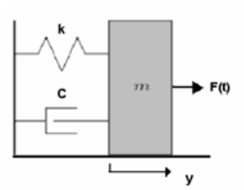
Damping Ratio of System $[\zeta]$

Defn: A measure of system damping – a measure of a system's ability to absorb or dissipate energy.

Figliola and Beasley, Theory and Design for Mechanical Measurements



We define the damping ratio for a system as:

$$\zeta = \frac{a_1}{2\sqrt{a_0 \cdot a_2}}$$

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Ringing Frequency of System $[\omega_d]$

Defn: The frequency of free oscillations of a damped system. A function of natural frequency and damping ratio

Figliola and Beasley, *Theory and Design for Mechanical Measurements*

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- Independent of input signal

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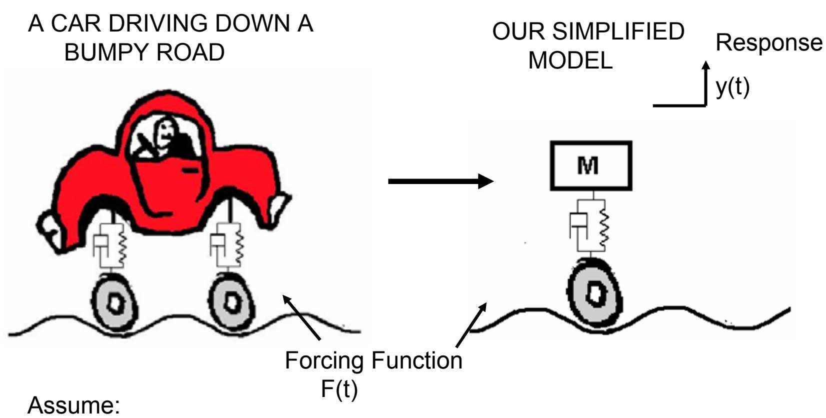
Harmonic Forcing Function



Assume:

- Sinusoidal road surface forcing function
- Spacing of bumps = wheel spacing
- Single degree of freedom system

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Harmonic Forcing Function

Mechanical Vibration: How mechanical systems respond to forcing function inputs.

Consider an everyday example – the motor vehicle.

A wide range of different inputs can cause vibrations in motor vehicles.

Wind	Engine Combustion
Road surface	Mechanical Imbalance
Engine Fan	Misalignment

All vibrations experienced by the driver and other occupants are the **result of mechanical dissipation of energy in response** to some forcing function input.

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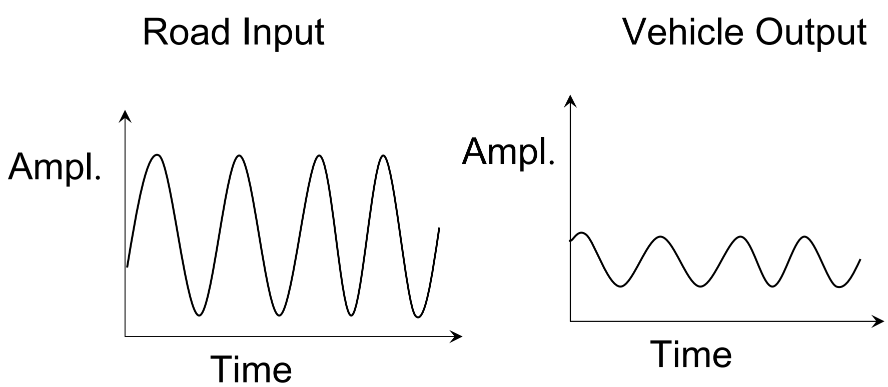
Response to Harmonic Excitation



Evaluation of these plots reveals two important quantities – gain and phase shift.

What contributes to these changes and how can we predict them?

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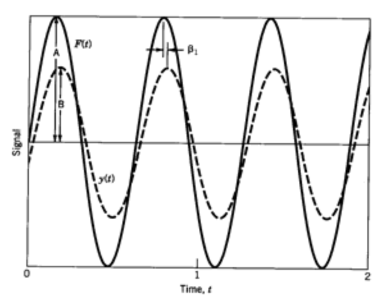
Harmonic Excitation

- Harmonically excited 2^{nd} Order System Response to Fluctuating Inputs

- Example of Harmonic excitation:

Mechanical Vibration



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2^{nd} Order Response

Sine Function Input

- Response of 2^{nd} order system to F(t)=A sin ωt

$$y(t) = y_h + \frac{\{KA \sin[wt + \varphi(w)]\}}{\{[1-(w/w_n)^2]^2 + (2\zeta w/w_n)^2\}^{1/2}}$$

- Frequency dependent phase shift

$$\varphi(w) = -\tan^{-1}(2\zeta w/w_n)/(1-(w/w_n)^2)$$

Exact form of y_h depends on (ζ) damping ratio

Note: h = homogeneous solution

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Harmonic Excitation

Steady State Response

- $Y_{steady}(t)=\beta(\omega)\sin[\omega t+\varphi(\omega)]$
- Amplitude:

  $B(\omega)=KA/\{[1-(\omega/\omega_n)^2]^2+(2\zeta\,\omega/\omega_n)^2\}^{1/2}$
- The amplitude of the output signal from a second-order measurement system is frequency dependent.

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Harmonic Excitation

Magnitude Ratio

- Magnitude Ratio:

$m(\omega) = B/KA$

- ω_n is a function of the measurement system
- ω is a function of the input signal

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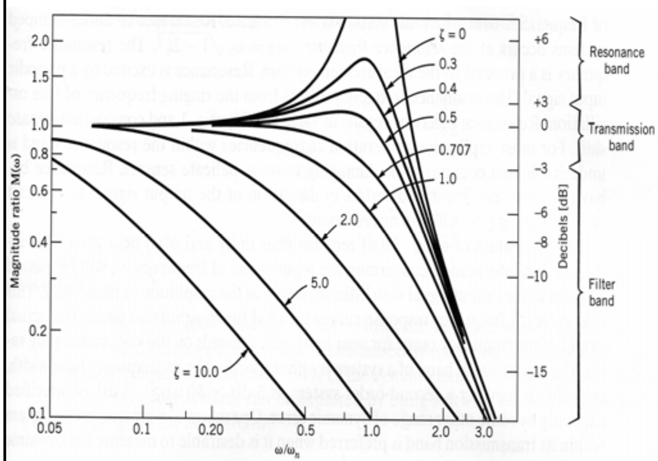
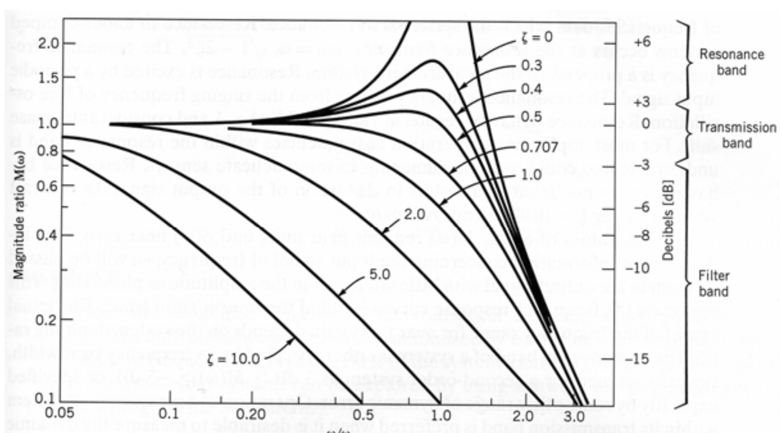
Harmonic Excitation



- Fig 3.16 and Fig 3.17 in text demonstrate magnitude and phase as functions of ω/ω_n
- In ideal system, $m(\omega)=1.0$ and $\varphi(\omega)=0$
- In general, as ω/ω_n gets large, $m(\omega)\rightarrow 0$ and $\varphi(\omega)\rightarrow -\pi$

Magnitude Ratio



OBSERVATIONS

• For low damping values, the amplitude is almost constant up to a frequency ratio of about 0.3 → Good linearity before frequency ratio of 0.3

• For large damping values (overdamped case), the amplitude is reduced substantially

• More on p 94 of Figliola

Harmonic Excitation

Resonance Frequency

$$W_r = W_n\sqrt{1 - 2\zeta^2}$$

- It is a property of MS, operating near resonance frequency. It can damage or distort either the data or the instrument.
- When $\omega = \omega_n$, $m(\omega) \rightarrow \infty$, and $\varphi(\omega) \rightarrow -\pi$, it occurs for the underdamped system $\zeta = 0$. It is called the resonance band.
- Systems with damping $\zeta > 0.7$ do not resonate.

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Harmonic Excitation

Resonance Frequency cont:

- At low ω/ω_n, $m(\omega) \approx 1$ and $\varphi(\omega) \approx 0$. This is called the transmission band, which is defined by

  $3\text{dB} \geq m(\omega) \geq -3\text{dB}$.
  - Here we have a representation of the dynamic signal content.
- At large ω/ω_n, $m(\omega) \rightarrow 0$, which is called the filter band
  - $m(\omega) \leq -3\text{dB}$
  - Here we lose high frequency signal content, which is good only if you want low frequency information!

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Filtering Effect of Magnitude Ratio

All mechanical systems act as low pass filters for two reasons.

1. High frequencies require higher speeds to reach the same amplitudes as lower frequencies
2. All machines have a maximum velocity (due to inertia). Once the maximum velocity is reached, higher frequencies can only be reached by reducing the amplitude.

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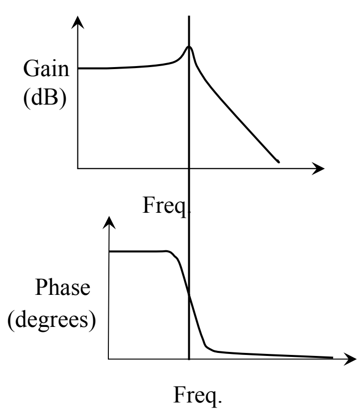
Filtering Effect of Magnitude Ratio

As the frequency increases the gain initially increases (until natural frequency) and then decreases (after natural frequency).



While low frequency inputs are passed through the system, high frequency inputs are attenuated.

Such a system is called a **low pass filter**.

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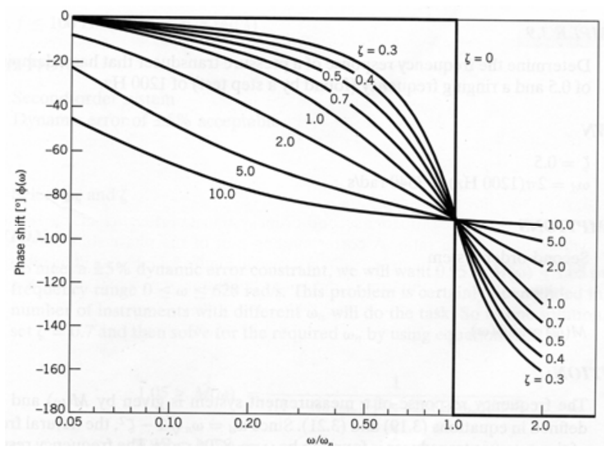
Phase Shift

The phase shift is the change in the position of the output vibration signal relative to the input vibration signal.



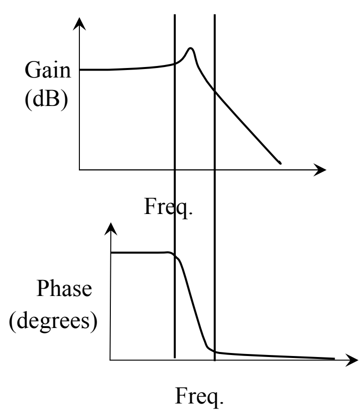
OBSERVATIONS

- The phase shift characteristics are a strong function of the damping ratio for all frequencies
- The frequency shift reaches a maximum of –180° at higher frequencies (above the natural frequency).

Mechanical Resonance

Defn: The frequency at which the magnitude ratio reaches a maximum value greater than unity.

Figliola and Beasley, Theory and Design for Mechanical Measurements



Systems natural frequency (resonance) occurs when the phase shift is exactly -90°

For underdamped systems we

observe a dramatic increase in gain.

For frequencies above resonance the gain decreases as the phase shift approaches -180 deg.

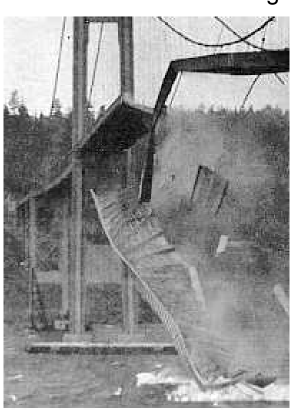
Tacoma Narrows Bridge

- The old Tacoma Narrows bridge was named *Galloping Gertie* after its completion in July 1940 because it vibrated violently in some wind conditions
- Crossing it was like a roller-coaster ride, and Gertie was quite popular.
- November 7, 1940, a day high winds, Gertie took on a 30-hertz *transverse* vibration with an amplitude of 1½ ***feet!***

Galloping Gertie After The Storm

- Wind-tunnel testing of bridge designs along with considerations for resonant behavior are now used to insure against similar disasters.

Old Tacoma Narrows Bridge



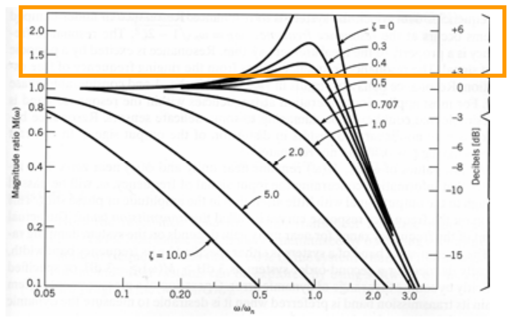
New Narrows Bridge!

Operating in the Resonance Band

The Resonance Frequency for underdamped systems:

$$\omega_R = \omega_n \sqrt{1-2\zeta^2}$$

Resonance Band



- Systems with damping ratio greater than 0.7 do not resonate
- Resonance is excited by a periodic input signal
- The peak gain occurs slightly below the system resonance due to damping
- Operating an underdamped systems at resonance can cause serious damage – Tacoma Narrows bridge

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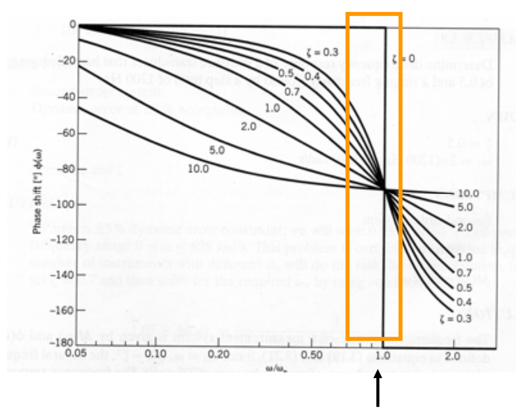
Phase Shift in the Resonance Band

Resonance Band



$$\omega / \omega_n = 1$$

OBSERVATIONS

- Phase shift jumps to –π
- The lower the damping ratio, the more sudden the jump

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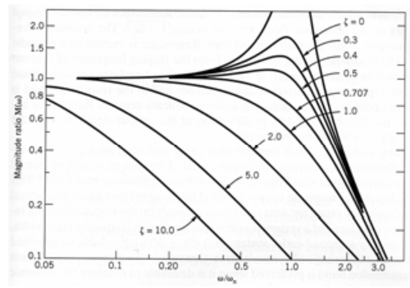
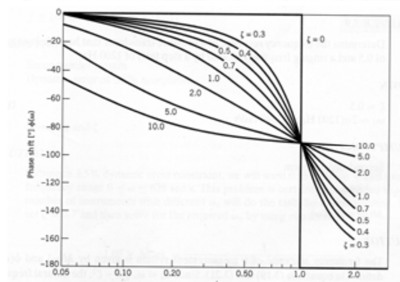
Pragmatic Interpretation

- Ideally system should have:
  - Linear frequency response across all ranges
  - With zero phase shift
- Real world
  - We must use an instrument over it's linear range where
    - Response is flat
    - Phase shift predictable
  - Some compensation is also possible using electronic methods





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Theoretical 2^{nd} Order Response

Steady state response to sinusoid input

$$y_{steady}(t) = B(\omega)\sin[\omega t + \Phi(\omega)]$$

$B(\omega)$ is Amplitude

Frequency Dependent Phase Shift

$$\Phi(\omega) = -\tan^{-1}\frac{2\cdot\zeta\cdot(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

As *ω/ω_n becomes large:*

$\Phi(\omega)$ approaches π

Frequency Magnitude Ratio

$$M(\omega) = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

As *ω/ω_n becomes large:*

$M(\omega)$ approaches zero

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Multiple-Function Inputs

- When models are used that are linear, ordinary differential equations subjected to inputs that are linear in terms of the dependent variable, the principle of superposition of linear systems will apply to the solution of these equations.

55

Principle of Superposition

- The theory of superposition states that a linear combination of input signals applied to a linear measurement system produces an output signal that is simply the linear addition of the separate output signals that would result if each input term had been applied separately.

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Principle of Superposition

- The forcing function of a form:

$$F(t) = A_0 + \sum_{i=1}^{\infty} (A_i \sin \omega_i t)$$

is applied to a system, then the combined steady response will have the form:

$$KA_0 + \sum_{n=1}^{\infty} B(\omega_i) \sin[\omega_i t + \phi(\omega_i)]$$

Where $B(\omega_i)=KA_iM(\omega_i)$

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Coupled Systems

- When a measurement system consists of more than one instrument, the measurement system behavior can become more complicated.
- As instruments in each stage of the system are connected, the output from one stage becomes the input to the next stage and so forth.

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Coupled Systems

- Such measurement systems will have an output response to the original input signal that is some combination of the individual instrument responses to the input.
- The system concepts of zero-, first-, and second-order systems studied previously can be used for a case-by-case study of the coupled measurement system.
- This is done by considering the input to each stage of the measurement system as the output of the previous stage.

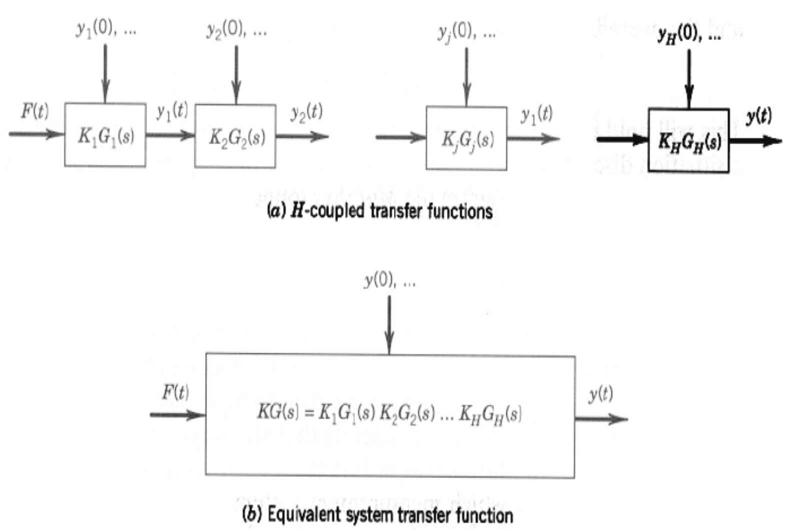

Figure 3.24 Coupled systems, describing the system transfer function.

Coupled Systems

- The previous slide depicts a measurement system consisting of H interconnected devices, j = 1, 2, …, H, each device described by a linear system model.

Coupled Systems

- The overall transfer function of the combined system, G(s), will be the product of the transfer functions of each of the individual devices, $G_j(s)$, such that:

  $KG(s) = K_1G_1(s)K_2G_2(s)\ldots K_HG_H(s)$
- The overall system static sensitivity is described by:

  $K = K_1K_2K_3\ldots K_H$

Coupled Systems

- The overall system magnitude ratio will be the product:

  $M(\omega) = M_1(\omega)M_2(\omega)\ldots M_H(\omega)$
- The overall system phase shift will be the sum:

  $\phi(\omega) = \phi_1(\omega) + \phi_2(\omega) + \ldots + \phi_H(\omega)$

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Probability And Statistics

Probability And Statistics

Engineering measurements taken repeatedly under seemingly ideal conditions will normally show variability.

Measurement system

- Resolution
- Repeatability

Measurement procedure and technique

- Repeatability

Probability And Statistics

Measured variable

- Temporal variation
- Spatial variation

We want:

1. A single representative value that best characterizes the average of the data set.
2. A measure of the variation in a measured data set.

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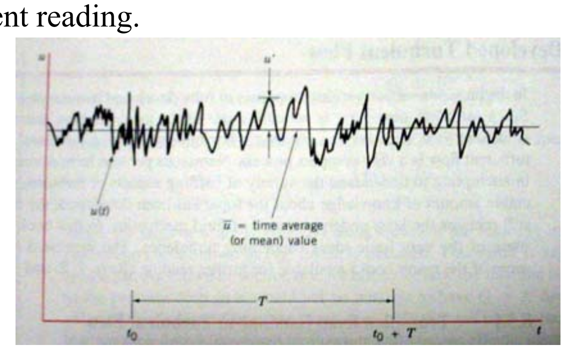
Random Variables

Say you have a velocity probe that you can put into a turbulent flow. You know that turbulent flows are characterized by random fluctuations in velocity. So, even though you may have the wind tunnel speed fixed, and nothing else is changing, every time you sample the velocity signal, you get a different reading.



There are three things you need to describe a random variable statistically: 1) The average value, 2) some description of the size of the variations and 3) what type of distribution.

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Probability And Statistics

We define *x'* to be the true mean value of the random variable *x*. With a finite number of samples, we will never know the value of *x'* exactly, but we will learn ways to estimate it.

True value is represented by $X' = \overline{X} \pm u_x$

$\overline{X}$ = most probable estimate of X'

u_x = confidence interval at probability P% which is based on estimates of precision and bias error.

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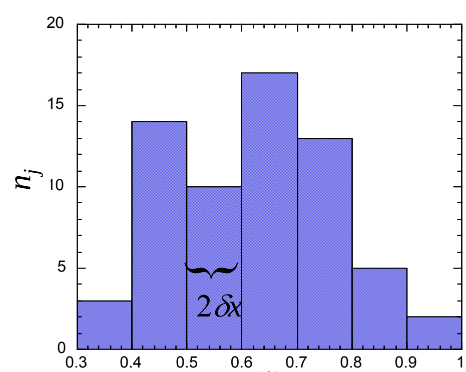
Random Variables

We are going to concentrate on random variables that have what is known as central tendency, which means that if you take a bunch of samples, many of them will collect near a single value. You can see central tendency if you plot the data in question on a histogram.



$N=64$
$K = 7$

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Probability Density Function

If we have a histogram plot, and we let the number of data points go to infinity while the size of the bins goes to zero, we get a "probability density function"

$$p(x) = \lim_{N \to \infty, \delta x \to 0} \frac{n_j}{N(2\delta x)}$$

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Probability And Statistics

Probability Density Function

Central Tendency – says that there is one central value about which all other values tend to be scattered.

Probability Density – the frequency with which the measured variable takes on a value within a given interval.

The region where observations tend to gather is around the central value.

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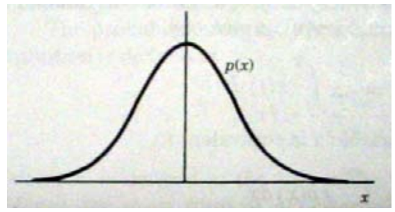
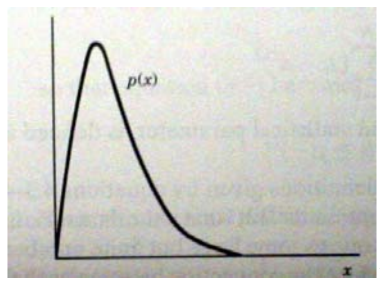
Probability Density Functions Examples: (see Table 4.2)

- Normal (Gaussian, bell curve) - Describes almost anything real if you take enough data



- Poisson - Describes events occurring randomly in time (e.g. radioactive decay)



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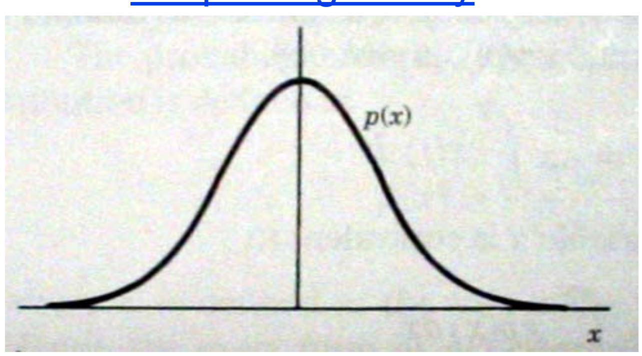
For a Normal Distribution the pdf is given by



$$p(x) = \frac{1}{\sigma (2\pi)^{1/2}} \exp\left[-\frac{1}{2} \frac{(x - x')^2}{\sigma^2} \right]$$

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Mean and Variance

- No matter which distribution you have, the mean value (central tendency) is given by

$$x' = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)$$

- And the variance is

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T [x(t) - x']^2 dt$$

- For infinite discrete series, these are

$$x' = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i \qquad \sigma^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} [x_i - x']^2$$

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Mean and Variance

- A fundamental difficulty arises in the definitions given by the equations earlier, in that they assume an infinite number of measurements
- What if the data set is finite ?
- We will look into the connection between probability and statistics and then into the practical treatment of finite sets of data

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Probability & Statistics

- Probability theory examines the properties of random variables, using the ideas of *random variables, probability* and *probability distributions.*
- Statistical measurement theory (and practice) uses probability theory to answer concrete questions about accuracy limits, whether two samples belong to the same population, etc.
- *"The analysis of data inevitably involves some trafficking with the field of* ***statistics****, that gray area which is not quite a branch of mathematics – and just as surely not quite a branch of science."* [H. Press *et. al.*, *Numerical Recipes*, *Cambridge Univ. Press, Chap. 14*]

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Probability & Statistics

- To find the interval in which the measurand value will fall under given experimental conditions we need to sort out few background ideas first
- A good start is to clarify the relationship between probability theory and statistical measurement theory

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Probability And Statistics

Plotting Histograms

- The abscissa is divided in K small intervals between the minimum and maximum values.
- The abscissa will be divided between the maximum and minimum measured values of x into K small intervals.
- Let the number of times, nj, that a measured value assumes a value within an interval defined by $x-\delta x \leq x < x+\delta x$ be plotted on the ordinate.
- For small N, K should be chosen so that $n_j \geq 5$ for at least one interval.

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Probability And Statistics

Plotting Histograms

- For N>40 ; $K=1.87(N-1)^{0.40}+1$
- The histogram displays the tendency and density.
- If the y-axis is normalized by dividing n_j/N, a frequency distribution results.

See Example 4.1 page 112

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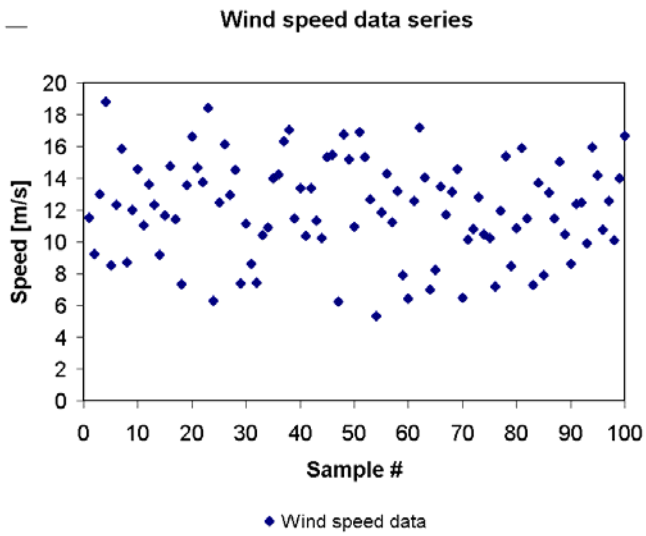
Having a set of repeated measurement data, we want to know how to:

1. Find the best estimate for the measured variable (the *measurand*)
2. Find the best estimate for the measurand *variability*
3. Find the *interval* in which the measurand value will fall under given experimental conditions

Example: Consider the Wind speed data:

Sample #	Wind speed data	Running average	Sample #	Wind speed data	Running average	Sample #	Wind speed data	Running average	Sample #	Wind speed data	Running average
1	11.5365	11.5365	26	16.1416	12.6104	51	16.9181	12.5474	76	7.1871	12.0829
2	9.2332	10.3849	27	12.9392	12.6226	52	15.3221	12.6008	77	11.9689	12.0814
3	12.9846	11.2515	28	14.5035	12.6898	53	12.6510	12.6017	78	15.3700	12.1236
4	18.7957	13.1375	29	7.3669	12.5062	54	5.3269	12.4670	79	8.4892	12.0776
5	8.5051	12.2110	30	11.1278	12.4603	55	11.8757	12.4563	80	10.8533	12.0623
6	12.3544	12.2349	31	8.6197	12.3364	56	14.2836	12.4889	81	15.8850	12.1095
7	15.8414	12.7501	32	7.4469	12.1836	57	11.2585	12.4673	82	11.4947	12.1020
8	8.7258	12.2471	33	10.4165	12.1300	58	13.1818	12.4796	83	7.3016	12.0441
9	11.9886	12.2184	34	10.9251	12.0946	59	7.8915	12.4019	84	13.7059	12.0639
10	14.5874	12.4553	35	14.0136	12.1494	60	6.4132	12.3020	85	7.8833	12.0147
11	11.0471	12.3272	36	14.2179	12.2069	61	12.5804	12.3066	86	13.0810	12.0271
12	13.6365	12.4363	37	16.3352	12.3185	62	17.2129	12.3857	87	11.4623	12.0206
13	12.3296	12.4281	38	17.0296	12.4424	63	14.0680	12.4124	88	15.0626	12.0552
14	9.1848	12.1965	39	11.4896	12.4180	64	7.0001	12.3279	89	10.4630	12.0373
15	11.6655	12.1611	40	13.3769	12.4420	65	8.2330	12.2649	90	8.6416	11.9996
16	14.7416	12.3224	41	10.3996	12.3922	66	13.4989	12.2836	91	12.3941	12.0039
17	11.4189	12.2692	42	13.3815	12.4157	67	11.7140	12.2751	92	12.4995	12.0093
18	7.3487	11.9958	43	11.3415	12.3907	68	13.1620	12.2881	93	9.9212	11.9869
19	13.5740	12.0789	44	10.2202	12.3414	69	14.5516	12.3209	94	15.9513	12.0290
20	16.6338	12.3067	45	15.3365	12.4080	70	6.4819	12.2375	95	14.1887	12.0518
21	14.6856	12.4199	46	15.4909	12.4750	71	10.1296	12.2078	96	10.7504	12.0382
22	13.7430	12.4801	47	6.2576	12.3427	72	10.8048	12.1883	97	12.5755	12.0437
23	18.4066	12.7378	48	16.7661	12.4348	73	12.8268	12.1971	98	10.0896	12.0238
24	6.2973	12.4694	49	15.1833	12.4909	74	10.4948	12.1741	99	14.0007	12.0438
25	12.4632	12.4692	50	10.9444	12.4600	75	10.2319	12.1482	100	16.6868	12.0902

Let's plot them first:



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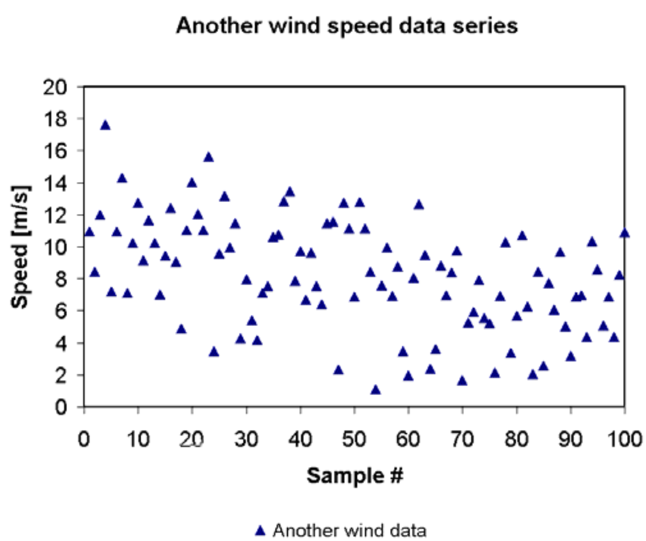
Look at another data set:



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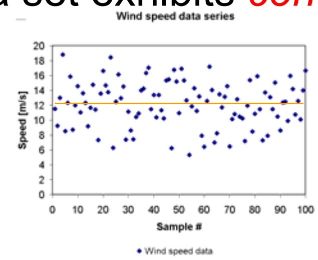
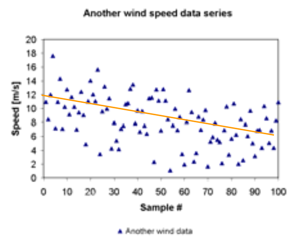
What can we conclude?

- ➢ First data set exhibits *central tendency*



Second data set exhibits *trend*



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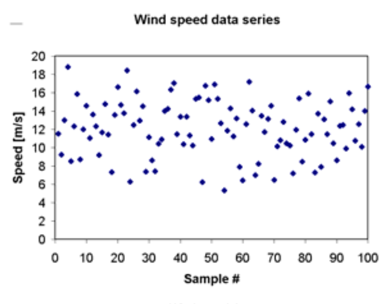
What else?



On the first set:

- ➢ The data points seem to oscillate around 12 or 13 or close to that
- ➢ The data points lie between 6 and 19 approximately

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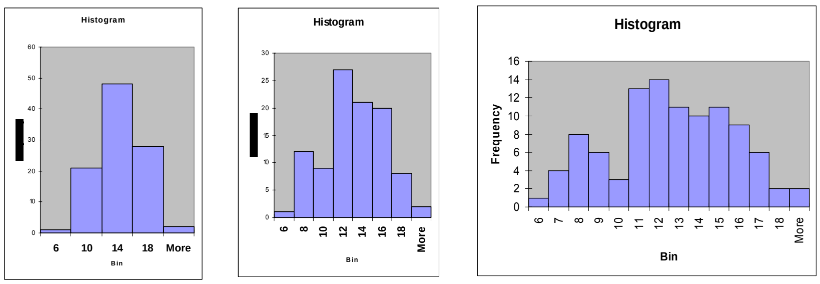
Next, we can make a histogram



Which one is the right one?

Number of bins, $K = 1.87(N-1)^{0.4} + 1$ (Eq. 4.2) pg. 111

For our data, $N = 100$, $K = 13$

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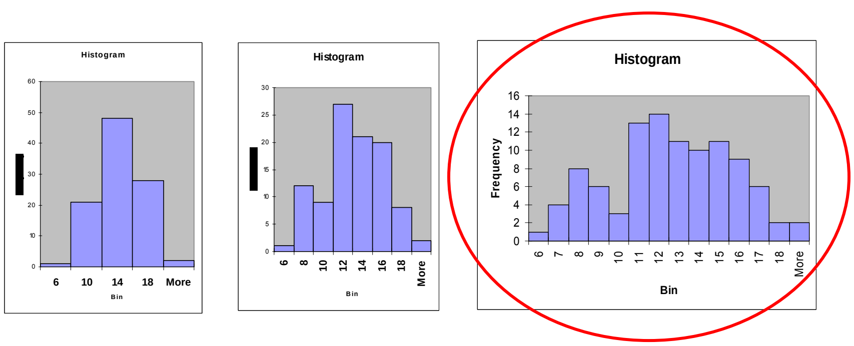
Next, we can make a histogram



Which one is the right one?

Number of bins, $K = 1.87(N-1)^{0.4} + 1$ (Eq. 4.2) pg. 111

For our data, $N = 100$, $K = 13$

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Let’s have a look at yet another data set:

Sample #	Value	Running average	Sample #	Value	Running average	Sample #	Value	Running average	Sample #	Value	Running average
1	3	3.0000	26	3	4.6538	51	0	4.8431	76	6	4.8158
2	1	2.0000	27	8	4.7778	52	5	4.8462	77	2	4.7792
3	4	2.6667	28	3	4.7143	53	8	4.9057	78	0	4.7179
4	1	2.2500	29	2	4.6207	54	2	4.8519	79	8	4.7595
5	5	2.8000	30	7	4.7000	55	0	4.7636	80	9	4.8125
6	9	3.8333	31	9	4.8387	56	9	4.8393	81	9	4.8642
7	2	3.5714	32	5	4.8438	57	7	4.8772	82	8	4.9024
8	6	3.8750	33	0	4.6970	58	4	4.8621	83	6	4.9157
9	5	4.0000	34	2	4.6176	59	9	4.9322	84	2	4.8810
10	3	3.9000	35	8	4.7143	60	4	4.9167	85	8	4.9176
11	5	4.0000	36	8	4.8056	61	4	4.9016	86	0	4.8605
12	8	4.3333	37	4	4.7838	62	5	4.9032	87	3	4.8391
13	9	4.6923	38	1	4.6842	63	9	4.9683	88	4	4.8295
14	7	4.8571	39	9	4.7949	64	2	4.9219	89	8	4.8652
15	9	5.1333	40	7	4.8500	65	3	4.8923	90	2	4.8333
16	3	5.0000	41	1	4.7561	66	0	4.8182	91	5	4.8352
17	2	4.8235	42	6	4.7857	67	7	4.8507	92	3	4.8152
18	3	4.7222	43	9	4.8837	68	8	4.8971	93	4	4.8065
19	8	4.8947	44	3	4.8409	69	1	4.8406	94	2	4.7766
20	4	4.8500	45	9	4.9333	70	6	4.8571	95	1	4.7368
21	6	4.9048	46	9	5.0217	71	4	4.8451	96	1	4.6979
22	2	4.7727	47	3	4.9787	72	0	4.7778	97	7	4.7216
23	6	4.8261	48	7	5.0208	73	6	4.7945	98	0	4.6735
24	4	4.7917	49	5	5.0204	74	2	4.7568	99	6	4.6869
25	3	4.7200	50	1	4.9400	75	8	4.8000	100	7	4.7100

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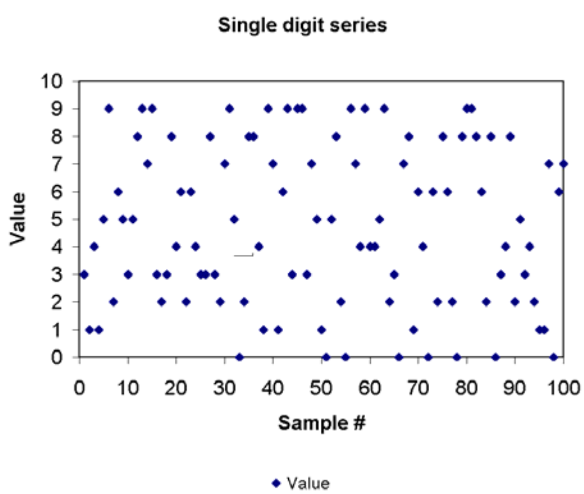
And plot the data…



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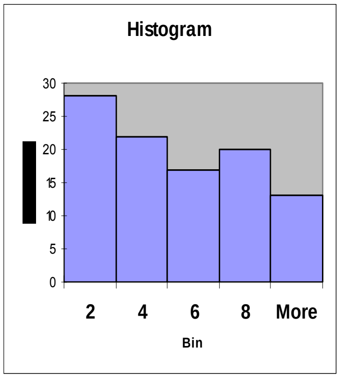
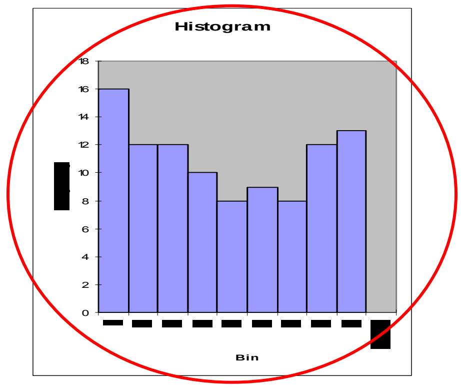
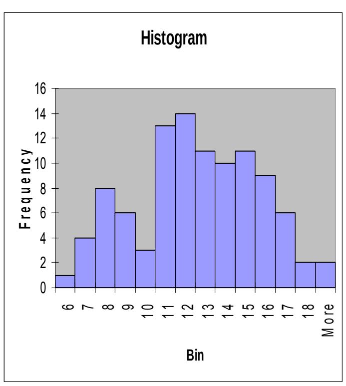
And make a histogram:

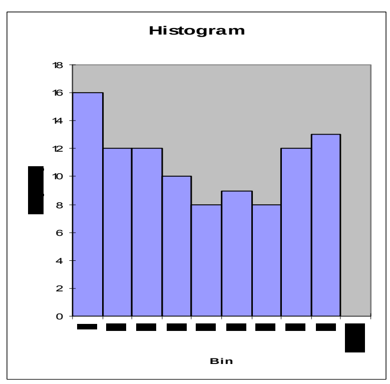




Let's compare the two histograms



Wind speed data



Second data set

Answer to Question 1:

1. To find the best estimate for the measured variable (the *measurand*)

   Use the mean value!

$$\bar{x} = \frac{1}{N}\sum_{1}^{N} x_i$$

Answer to Question 2:

2. To find the best estimate for the measurand *variability*

Use the Sample variance $S_x^2 = \frac{1}{N-1}\sum_{1}^{N}\left(x_i - \bar{x}\right)^2$

Or the Sample standard deviation $S_x = \sqrt{S_x^2}$

How do we add up sample means, variances and standard deviations?

Let $x = a + b$ Then,

$\bar{x} = \bar{a} + \bar{b}$ Means add up

$S_x^2 = S_a^2 + S_b^2$ Variances add up

$S_x = \sqrt{S_a^2 + S_b^2}$ Standard deviations don't simply add up

Provided that a and b are uncorrelated!

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A bathroom scale example

- A series of 100 weight measurements were performed on a (badly tuned) bathroom scale. A weight of 50 lb was repeatedly measured by a class in 1996. 10 groups made 10 measurements each.
- The raw data with running average is shown in the following figure.

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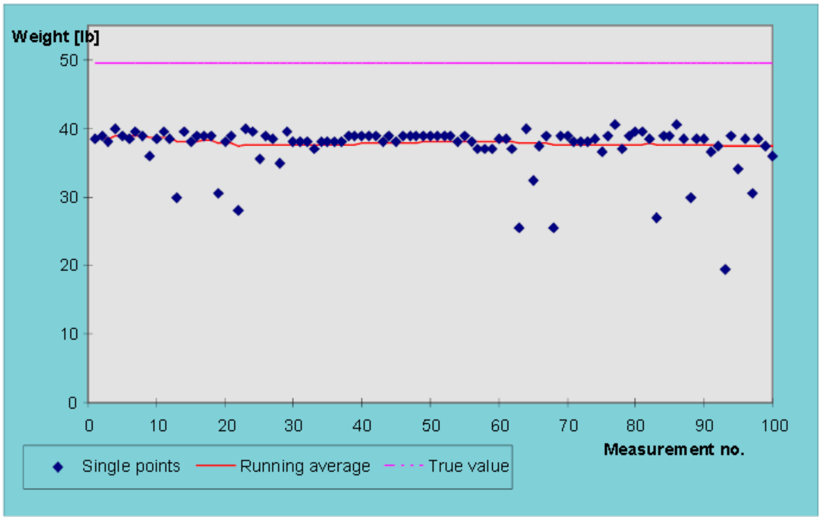

A series of 100 weight measurements

(Running average at point *i* is an average of points from 1 to *i*)

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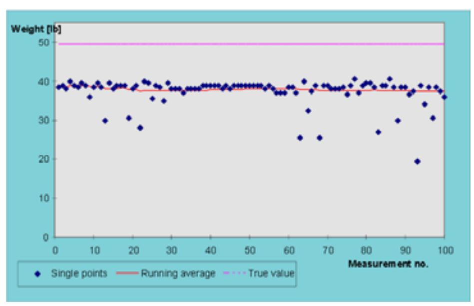
Some initial observations



- The data scatter varies from group to group (i.e. the ranges 31-40, 41-50 and 51-60 are notably more uniform).
- Although the data scatter is considerable, the running average quickly converges to a stable value (central tendency).
- Data points tend to cluster closer to the upper limit of the data range.
- The true value is outside the data scatter, indicating heavy bias.
- The bias is negative.

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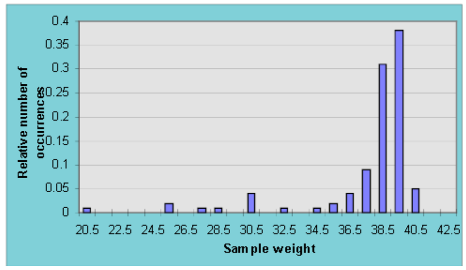
Next we examine the histogram



- The histogram confirms that the data are *skewed* towards upper limit.
- The histogram gives an intuitive idea of a *probability distribution*, but we cannot refine it much more.

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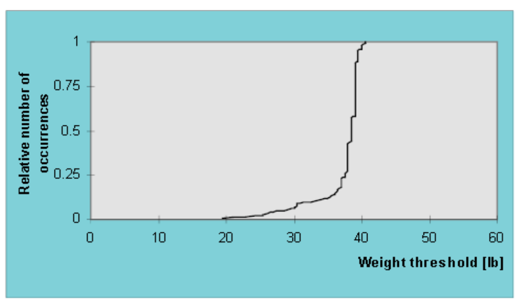
A different route



Sample cumulative distribution

- The sample cumulative distribution is much smoother and does not require the selection of bins and the distribution of data into them. A smooth curve is easy to fit through the sample (staircase) curve.

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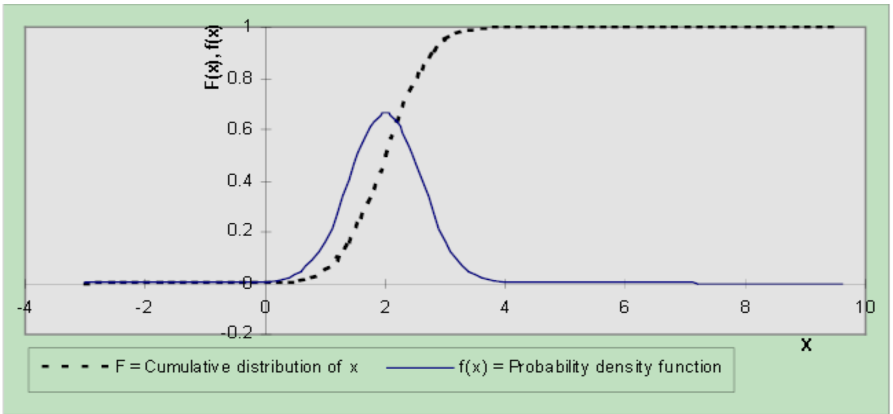
The probability distribution



$$f(x) = \frac{dF(x)}{dx} \quad \text{and} \quad F(x) = \int_{-\infty}^{x} f(x)dx$$

$$P(a < x \le b) = F(b) - F(a) = \int_{a}^{b} f(x)dx$$

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The Probability Density Function

P(x) comes from the frequency distribution

$$P(x) = \lim_{N\to\infty, \delta x \to 0} (n_j)/(N(2\delta_x))$$

- Defines the probability that a measured variable might assume a particular value on any given observation, and also provides the central tendency
- The shape depends on the variable in consideration and its natural circumstances/processes.
- Plot histograms – compare to common distribution and then fit the parameter.
- Unifit is a good PC-based distribution fitting software.

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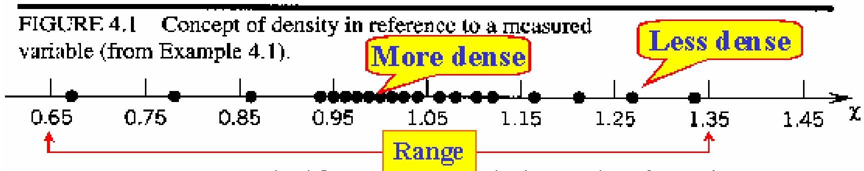
The Probability Density Function

TABLE 4.1 Sample of Variable x

i	x_i	i	x_i
1	0.98	11	1.02
2	1.07	12	1.26
3	0.86	13	1.08
4	1.16	14	1.02
5	0.96	15	0.94
6	0.68	16	1.11
7	1.34	17	0.99
8	1.04	18	0.78
9	1.21	19	1.06
10	0.86	20	0.96

FIGURE 4.1 Concept of density in reference to a measured variable (from Example 4.1).



K small intervals required for a viable statistical analysis is found from

$$\boxed{K = 1.87(N-1)^{0.40} + 1 \qquad n_i \geq 5 \text{ for at least one interval}} \quad (4.2)$$

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Infinite Statistics

- In a perfect world, we have an infinite number of data points. In reality, this is not the case, but for now, we assume it is.
- If we wish to know the probability of x taking on some range of values,we simply integrate the pdf over that range:

$$P(x'-\delta x \leq x \leq x'+\delta x) = \int_{x'-\delta x}^{x'+\delta x} p(x)\,dx$$

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Infinite Statistics

- If we know what type of distribution we have and we know x' and σ, then we know *p(x)* and we could perform the integral.
- We can make the integral easier by transforming the variables a little bit.
- Make $z_1 = (x_1 - x')/\sigma$ and $\beta = (x - x')/\sigma$.
- Put in the normal distribution for P(x) and get:

$$P(-z_1 \le \beta \le z_1) = \frac{1}{(2\pi)^{1/2}} \int_{-z_1}^{z_1} e^{-\beta^2/2} d\beta$$

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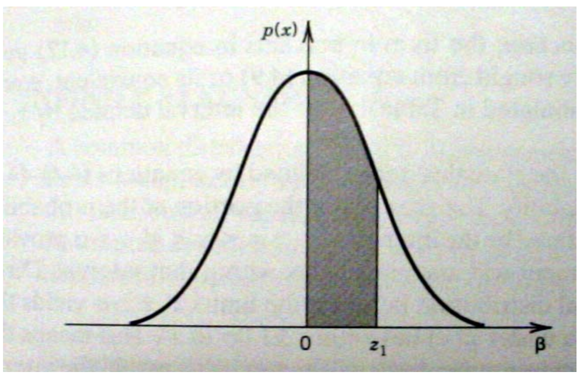
Infinite Statistics

The most common distribution is the normal or Gaussian distribution. This predicts that the distribution will be evenly distributed around the central tendency. ("bell curve")



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Infinite Statistics

$$P(-z_1 \le \beta \le z_1) = \frac{1}{(2\pi)^{1/2}} \int_{-z_1}^{z_1} e^{-\beta^2/2} d\beta$$

- Since the gaussian distribution is symmetric about x', we can write this as

$$P(-z_1 \le \beta \le z_1) = 2\left[\frac{1}{(2\pi)^{1/2}} \int_{0}^{z_1} e^{-\beta^2/2} d\beta\right]$$

- The term in the brackets is called the error function. You have probably seen it before, you certainly will again, and now you know why they call it that.

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Probability Density Function and Probability

$$p(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\frac{(x-x')^2}{\sigma^2}\right]$$

p(x)=dP/dx

$$P(x' - \delta x \le x \le x' + \delta x) = \int_{x'-\delta x}^{x'+\delta x} p(x)\, dx$$

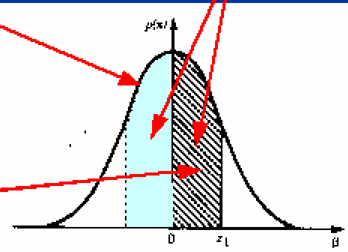
$$dx = \sigma\, d\beta \qquad (4.10)$$

4.9 becomes

$$P(-z_1 \le \beta \le z_1) = \frac{1}{(2\pi)^{1/2}} \int_{-z_1}^{z_1} e^{-\beta^2/2} d\beta \qquad (4.11)$$

al distribution, $p(x)$ is symmetrical about x', one can write

$$\frac{1}{(2\pi)^{1/2}} \int_{-z_1}^{z_1} e^{-\beta^2/2} d\beta = 2\boxed{\left[\frac{1}{(2\pi)^{1/2}} \int_0^{z_1} e^{-\beta^2/2} d\beta\right]} \qquad (4.12)$$



β=(*x*-*x'*)/σ = dim'less deviation   For *x*=*x'*, *β*=*0*

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Probability values for Normal Error Functions
Table 4.2

One-Sided Integral Solutions for $p(z_1) = \frac{1}{(2\pi)^{1/2}} \int_0^{z_1} e^{-\beta^2/2} d\beta$

$z_1 = \frac{x_1 - x'}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422

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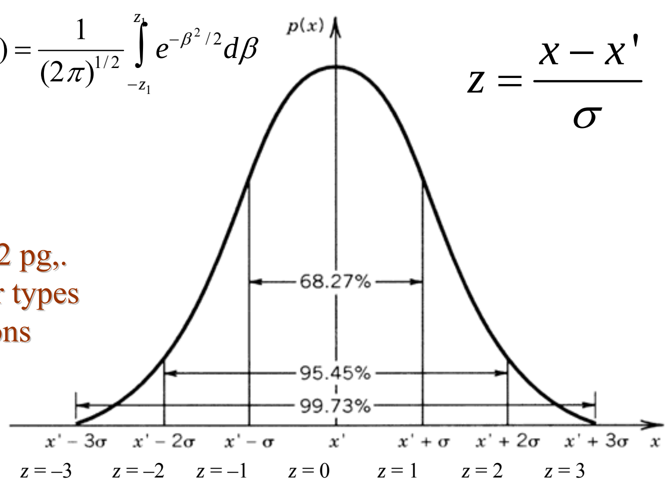
Normal or Gaussian Distribution

$$P(-z_1 \le \beta \le z_1) = \frac{1}{(2\pi)^{1/2}} \int_{-z_1}^{z_1} e^{-\beta^2/2} d\beta$$

$$z = \frac{x - x'}{\sigma}$$



See Table 4.2 pg,. 114 for other types of distributions

Note: p(x) is the "probability density of x"

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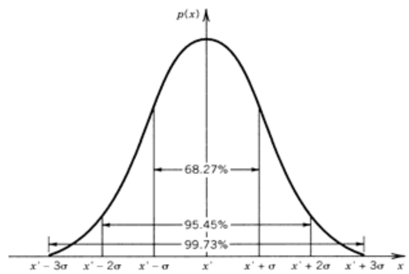
Gaussian Distribution Highlights

- We expect that measurements will show Gaussian distributed deviations due to random variations.
- ± One standard deviation contains 68.3% of data.
- ± Two standard deviations contain 95.5% of data.
- ± Three standard deviations contain 99.7% of data.



Normal or Gaussian Distribution

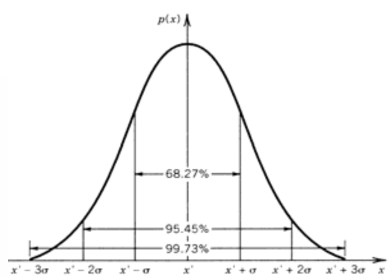
Solutions given for this in table 4.3

For z=1, 68.27% of observations within 1 standard deviation of x'

For z=2, 95.45%

For z=3, 99.73%

The values of z in table 4.3 can be used to predict probability of a unique value occurring in an infinite data set.



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Normal or Gaussian Distribution

- Pdf for Gaussian:

$$p(x) = 1/(v(2\pi)^{1/2} \exp\ [(-1/2)((x - x')^2/\sigma^2)]$$

x'= true mean of x ; σ^2 = true variance of x

- To find/predict the probability that a future measurement falls within some interval. Probability P(x) is the area under the curve between X'±δx on p(x)

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Normal Error Function Table

Table 4.3 Probability Values for Normal Error Function

One-Sided Integral Solutions for $p(z_1) = \frac{1}{(2\pi)^{1/2}} \int_0^{z_1} e^{-\beta^2/2} d\beta$

$z_1 = \frac{x_1 - x'}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633

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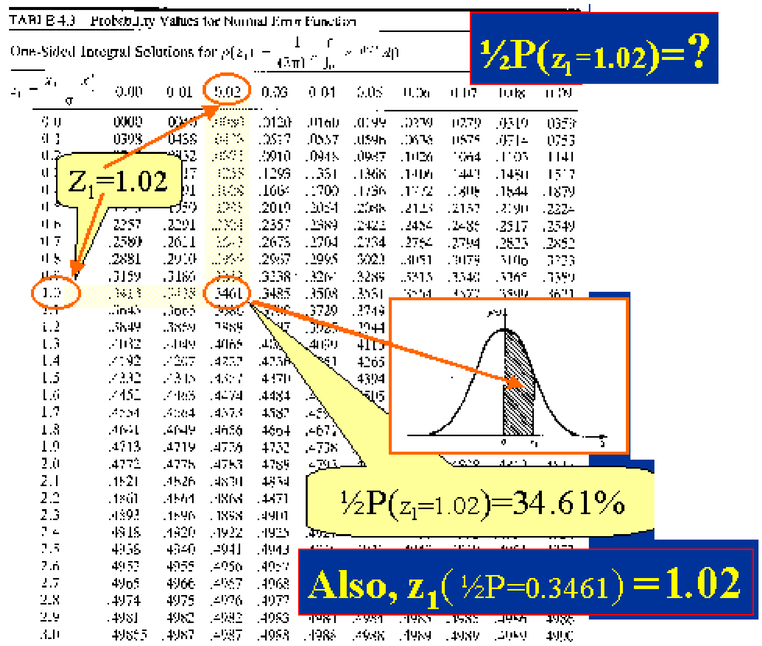
Normal or Gaussian Distribution



Finite Statistics

- When '*N*' is finite (less than infinity) all of the characteristics of a measured value may not be contained in '*N*' data points
- Statistical values obtained from finite data sets should be considered only a estimates of the true statistics
- Such statistics is called "Finite Statistics"
- Whereas the true behavior of a variable is described by its infinite statistics, finite statistics only describe the behavior of the finite data set

Finite Statistics

When $N << \infty$, we do not have true representation of the population. We have finite statistics which describe the sample and estimate the population.

Sample mean
(probable estimate of true mean)

$$\bar{x} = 1/N \sum_{i=1}^{N} x_i \qquad (4.14a)$$

Sample variance
(measurement of precision)

$$s_x^2 = 1/(N-1) \sum_{i=1}^{N} (x_i - \bar{x})^2 \qquad (4.14b)$$

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Finite Statistics

Deviation of x_i: $x_i - \bar{x}$

Sample standard deviation: $S_x = \sqrt{S_x^2}$

Degree of freedom: the number of samples less the central tendency measurement (N-1)

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t-estimate

- For a finite data set, we use the t-estimate instead of z, which was used in the infinite.
- One can state that (P%)

$$x_i = \bar{x} \pm t_{v,p} S_{\bar{x}} \qquad (4.15)$$

$\pm t_{v,p} S_{\bar{x}}$ represents the precision interval

P% = probability

v = degrees of freedom

x = sample mean

Table 4.4 gives "t" distribution

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Student-t Distribution – Table 4.4 in the text

Table 4.4 Student-*t* Distribution

v	t_{50}	t_{90}	t_{95}	t_{99}
1	1.000	6.314	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.182	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
∞	0.674	1.645	1.960	2.576

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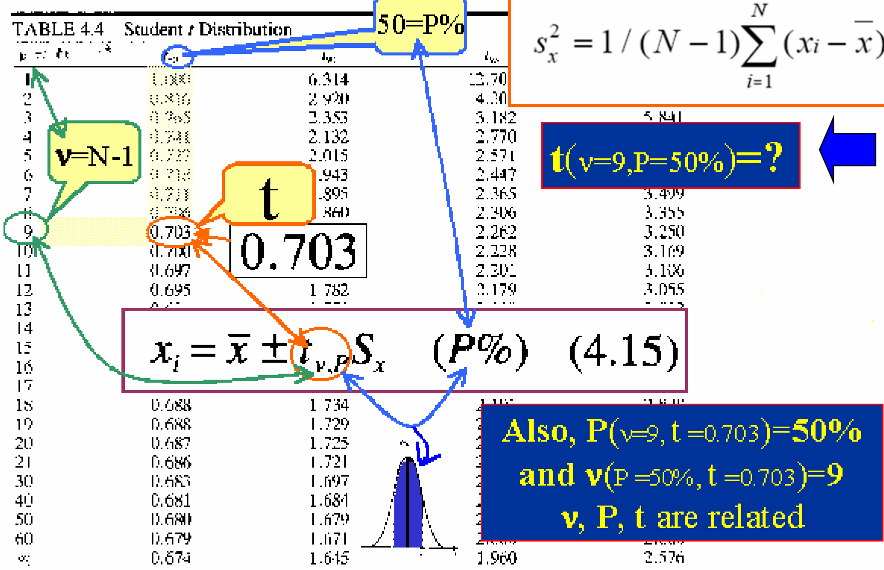
Finite Statistics
Student's t-distribution

$$\bar{x} = 1/N\sum_{i=1}^{N} x_i \qquad (4.14a)$$

$$s_x^2 = 1/(N-1)\sum_{i=1}^{N}(x_i - \bar{x})^2$$



If we know the true standard deviation

- We express the range of possible measurand values as a *confidence interval*, given at a certain *confidence level*:

$$x' = \bar{x} \pm u_x \qquad (P\%)$$

- If we know the true standard deviation, σ, then

$$x' = \bar{x} \pm z_{50}\sigma_{\bar{x}} = \bar{x} \pm 0.67\sigma_{\bar{x}} \qquad (50\%)$$

$$x' = \bar{x} \pm z_{95}\sigma_{\bar{x}} = \bar{x} \pm 1.96\sigma_{\bar{x}} \qquad (95\%)$$

$$x' = \bar{x} \pm z_{99}\sigma_{\bar{x}} = \bar{x} \pm 2.58\sigma_{\bar{x}} \qquad (99\%)$$

z_{95} is the value of z for which $P(0 < z \le z_{95}) = \dfrac{0.95}{2} = 0.475$

- The probability that the i th measured value of x will have a value between $x' \pm z_1\sigma$, is $2P(z_1) \times 100 = P\%$

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If we don't know the true standard deviation

- We use the sample standard deviation, S_x, but we pay the "penalty":

$$x' = \bar{x} \pm t_{\nu,50} S_{\bar{x}} = \bar{x} \pm 0.692 S_{\bar{x}} \qquad (50\%), \nu = 14$$
$$x' = \bar{x} \pm t_{\nu,95} S_{\bar{x}} = \bar{x} \pm 2.145 S_{\bar{x}} \qquad (95\%), \nu = 14$$
$$x' = \bar{x} \pm t_{\nu,99} S_{\bar{x}} = \bar{x} \pm 2.977 S_{\bar{x}} \qquad (99\%), \nu = 14$$

$t_{14,95}$ is the value of t for $\nu = 14$ and $P = 95\%$, table 4.4

Note: The figures above are an example for the sample size of $N = 15$ samples, with $\nu = N\text{-}1 = 14$ degrees of freedom.

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How do the true and sample standard deviations of the mean depend on the sample size?

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \; ; \qquad S_{\bar{x}} = \frac{S_x}{\sqrt{N}} \text{ , thus}$$

$$x' = \bar{x} \pm z_P \frac{\sigma_x}{\sqrt{N}} \text{ if } \sigma_x \text{ is known, or}$$

$$x' = \bar{x} \pm t_{N-1,P} \frac{S_x}{\sqrt{N}} \text{ if } \sigma_x \text{ is not known}$$

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Standard Deviation of the Means

If we measure a variable N times under fixed conditions, and replicate this M times, we will end up with slightly different means for each replication.

It can be shown that regardless of the form of the pdf of the individualized replication, the mean values themselves will be normally distributed.

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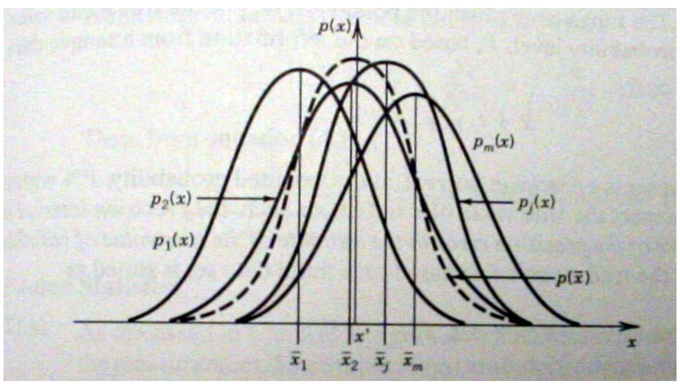
Standard Deviation of the Means



$$S_{\bar{x}} = \frac{S_x}{N^{1/2}}$$

$$x' = \bar{x} \pm t_{v,P} S_{\bar{x}} \qquad (P\%)$$

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Example for small samples

- A Micrometer is calibrated to eliminate bias errors
- Four independent measurements of shaft diameter are made: 25.04, 24.91, 24.98, 25.06 mm
- What is the uncertainty associated with the measurements in this application?
- Variations are partly due to the measurement process, partly due to real variation in the shaft diameter.

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Sample Statistics (Example)

$$\bar{x} = \frac{1}{N}\sum x_i = 24.9975 \text{ mm}$$

$$S_x = \sqrt{\frac{1}{N-1}\sum\left(x_i - \bar{x}\right)^2} = 0.0675 \text{ mm}$$

- Degrees of freedom, $\nu = N - 1 = 3$
- $t_{3,95}$ = 3.182 from Table 4.4
- Our uncertainty is about 1.5 times larger because of the small sample (3.182 *vs* 1.96)
- 1.96 is the value for $t_{3,95}$ at N = infinity

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Mean of four measurements

$$t_{3,95}\frac{S_x}{\sqrt{N}} = 3.182\frac{0.0675}{\sqrt{4}} = 0.107\text{ mm} \approx 0.11\text{ mm}$$

- The mean diameter of the shaft is 25.00 mm ±0.11 mm at 95% confidence level.

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An additional measurement

- If we now make a single measurement on another shaft and get 24.90 mm, then

$$t_{3,95}S_x = 3.182 \times 0.0675 = 0.215$$

- The diameter of that shaft at that location is

D = 24.90 mm ± 0.22 mm (95%).

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Pooled Statistics

- This section just states that if you perform M separate identical experiments, each consisting of N samples, you will get the same statistics as if you had a single experiment with M x N measurements (seems pretty obvious ?)

  - Samples that are grouped in a manner so as to determine a common set of statistics are called pooled.
  - If we have M replicates of variable x, with N repeated measurements producing data set $x_{i,j}$; i = 1 to N ; j = 1 to M

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Pooled Statistics

- Pooled mean of x:

$$\langle \bar{x} \rangle = 1/MN \sum_{j=1}^{M} \sum_{i=1}^{N} x_{ij}$$

- Pooled standard deviation of x:

$$\langle s_x \rangle = \sqrt{[1/(M(N-1)) \sum_{j=1}^{M} \sum_{i=1}^{N} (x_{ij} - \bar{x}_j)^2]}$$

$$= \sqrt{1/M \sum_{j=1}^{M} s_{x_j}^2}$$

(with v=M(N-1) degrees of freedom)

- Pooled standard deviation of means:

$$\langle s_{\bar{x}} \rangle = \langle s_x \rangle / (MN)^{1/2}$$

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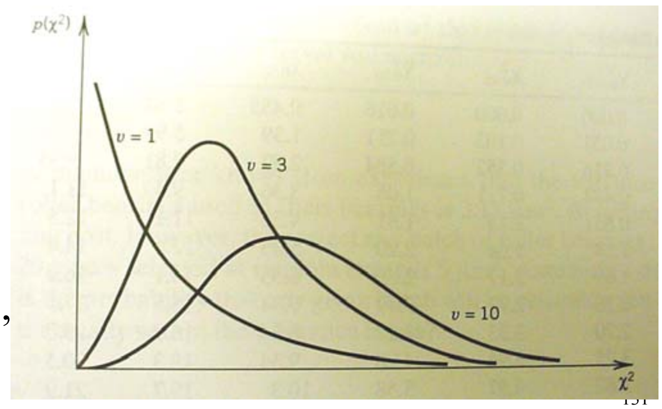
Chi-Squared Distribution (ki-squared)

If we were to repeat our N measurements a few times, we would compute a different estimate of the standard deviation S_x each time. (Remember that we said the same thing about the mean). There is a distribution (pdf) of the variance of measurements of a gaussian (normal) process, and it is called Chi-Squared.

$$\chi^2 = \nu S_x^2/\sigma^2,$$
$$\nu = N - 1$$



Chi-Squared Distribution

- Estimates the precision by which s^2_x predicts σ^2.
  - s^2_x = sample variance ; σ^2 = population variance
- If we plotted s_x for many sets having N samples each, we would generate the pdf for $P(\chi^2)$ (chi-squared)
- For a normal distribution chi-squared

$$\chi^2 = \nu(s^2_x/\sigma^2) \qquad \nu = N - 1$$

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Chi-Squared Distribution

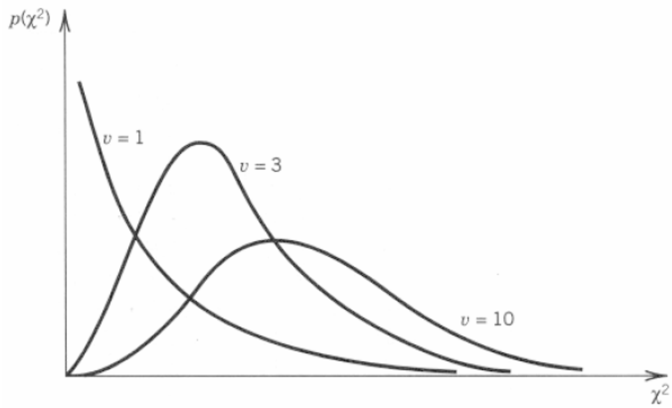

Figure 4.7 The χ^2 distribution with its dependency on degrees of freedom.

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Chi-Squared Distribution

Precision Interval in Sample Variance

- The precision interval for the sample variance can be formulated by the probability statement:

$$P(\chi^2_{1-\alpha/2} \le \chi^2 \le \chi^2_{\alpha/2}) = 1 - \alpha$$

with a probability of $P(\chi 2) = 1 - \alpha$;
α = level of significance

Combining:

$$P[\nu s_x^2 / \chi^2_{\alpha/2} \le \sigma^2 \le \nu s_x^2 / \chi^2_{1-\alpha/2}] = 1 - \alpha$$

For 95% precision interval by which s_x^2 estimates σ^2:

$$\nu s_x^2 / \chi^2_{.025} < \sigma^2 < \nu s_x^2 / \chi^2_{.975}$$

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Chi-Squared Distribution

Precision Interval in Sample Variance

- The χ^2 distribution estimates the discrepancy expected as a result of random chance.
- Values for χ^2_α are tabulated in Table 4.5 as a function of the degrees of freedom.
- The $P(\chi^2)$ value equals the area under $p(\chi^2)$ as measured from the left, and the α value is the area as measured from the right.
- The total area under $p(\chi^2)$ is equal to unity.

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Chi-Squared Distribution

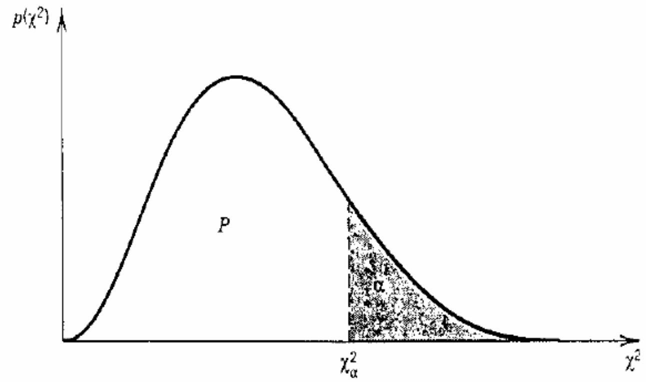

Figure 4.8 The χ^2 distribution as it relates to probability P and to the level of significance, $\alpha(= 1 - P)$.

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Chi-Squared Distribution

Values for χ^2_α

ν	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.50}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$
1	0.000	0.000	0.000	0.016	0.455	3.84	5.02	6.63
2	0.020	0.051	0.103	0.211	1.39	5.99	7.38	9.21
3	0.115	0.216	0.352	0.584	2.37	7.81	9.35	11.3
4	0.297	0.484	0.711	1.06	3.36	9.49	11.1	13.3
5	0.554	0.831	1.15	1.61	4.35	11.1	12.8	15.1
6	0.872	1.24	1.64	2.20	5.35	12.6	14.4	16.8
7	1.24	1.69	2.17	2.83	6.35	14.1	16.0	18.5
3	1.65	2.18	2.73	3.49	7.34	15.5	17.5	20.1
[illegible]	2.09	2.70	3.33	4.17	8.34	16.9	19.0	21.7
[illegible]	2.56	3.25	3.94	4.78	9.34	18.3	20.5	23.2
1	3.05	3.82	4.57	5.58	10.3	19.7	21.9	24.7
2	3.57	4.40	5.23	6.30	11.3	21.0	23.3	26.2
3	4.11	5.01	5.89	7.04	12.3	22.4	24.7	27.7
4	4.66	5.63	6.57	7.79	13.3	23.7	26.1	29.1
5	5.23	6.26	7.26	8.55	14.3	25.0	27.5	30.6
6	5.81	6.91	7.96	9.31	15.3	26.3	28.8	32.0
7	6.41	7.56	8.67	10.1	16.3	27.6	30.2	33.4
8	7.01	8.23	9.39	10.9	17.3	28.9	31.5	34.8
9	7.63	8.91	10.1	11.7	18.3	30.1	32.9	36.2
0	8.26	9.59	10.9	12.4	19.3	31.4	34.2	37.6
0	15.0	16.8	18.5	20.6	29.3	43.8	47.0	50.9
0	37.5	40.5	43.2	46.5	59.3	79.1	83.3	88.4

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Goodness of Fit Test

- We can use a chi-squared test to determine how good of a fit our selected pdf represents the actual distribution of data.
- The χ^2 test gives us the measure of error between the variation in the data set and variation predicted by the assumed pdf.
- Construct histogram of data and histogram from predicted pdf, where n_j is actual and n'_j is predicted number of occurrences per cell. Then:

$$\chi^2 = \sum_{j=1}^{k} (n_j - n'_j)^2 / n_j$$

For given d.o.f. the better fit gives lower χ^2

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Goodness of Fit Test

- The χ^2_α table, given in the previous slide, can be interpreted as a measure of the discrepancy expected as a result of random chance.
- For example, a value for α of 0.95 implies that 95% of the discrepancy between the histogram and the assumed distribution is due to random variation only.
- With $P(\chi^2) = 1 - \alpha$, this leaves only 5% of the discrepancy caused by a systematic tendency, such as different distribution.
- In general, a $P(\chi^2) < 0.05$ confers a very strong measure of a good fit to the assumed distribution, an unequivocal result.

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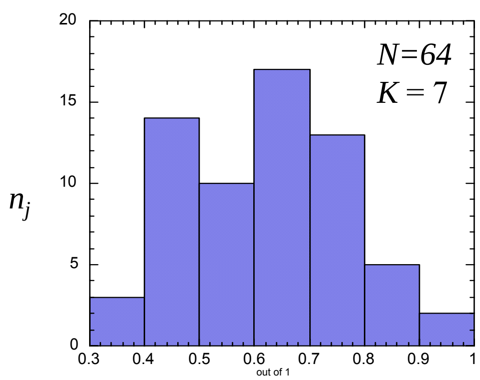
Goodness of Fit Test

We are going to assume that our data fits a normal (gaussian) distribution. If we have a set of data and we want to make sure this is a good fit, we use this test.



$$\chi^2 = \sum_j \frac{(n_j - n'_j)^2}{n'_j}$$

$j = 1, 2, \ldots K$

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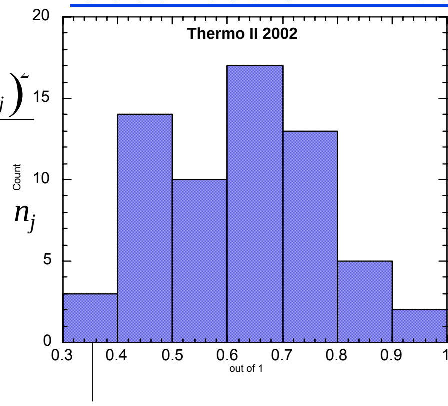
Goodness of Fit Test

$$\chi^2 = \sum_j \frac{(n_j - n'_j)^2}{n'_j}$$



N=64
K = 7

$j = 1 \ldots K$

$n'_1 = P\ (0.3 \le x \le 0.4) = P\ (0.3 \le x_i \le 0.62) - P\ (0.4 \le x_i \le 0.62)$
$= P(z_a) - P(z_b)$

Recall that $z_a = (x_a - x') / \sigma$
$= P(-2.26) - P(-1.55) = 0.4881 - 0.4394$
$= 0.0487$, or 4.9%

4.9% of 64 is 3.11, so $n_j - n'_j = -0.11$

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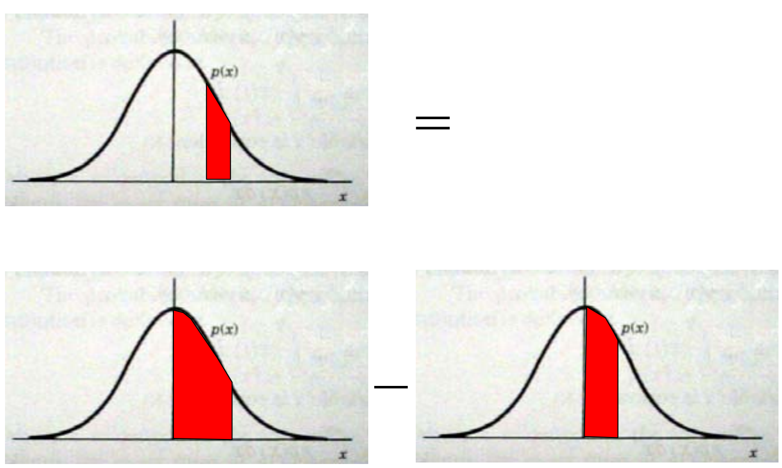
Probability in a band



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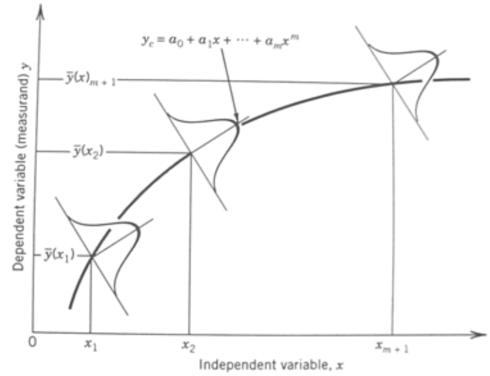
Regression Analysis



Used to determine the functional form of data. Often, we will know from some theory that $y = Ae^{-Bx}$, but we may not know what A and B are. A regression analysis can find the best guess for A and B. This type of analysis can be done for any $y = f(x)$ (your book incorrectly says that it is limited to polynomials).

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Least-Squares Regression

Generally, we will assume that our data represents some function $y = f(x; a, b, \ldots)$ where a and b are the coefficients to be determined. This technique tries to minimize (Least) the square of the difference (squares) between the data and the assumed function. It will calculate the values of a, b,…. that do this. If the assumed function is linear in a, b, …., then this method can do it in one shot. If not, it will iterate starting with initial guesses for all of the parameters.

I am not interested in you understanding the nuts and bolts of this technique. You can get the subroutines from Numerical Recipes or use a graphing package like Kaleidagraph. I do want you to understand the application and limits of the technique.

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Curve Fitting Examples

Number of Measurements Required

Again, the books discussion on this topic is confusing since it uses a t-estimator. There is nothing wrong with this approach, but it is perhaps easier to understand using infinite statistics. Say we have a measurement of some random data, and we want to know its mean with an error smaller than 5%. If we have some estimate of its standard deviation, then we know that

$$S_{\bar{x}} = \frac{S_x}{N^{1/2}}$$

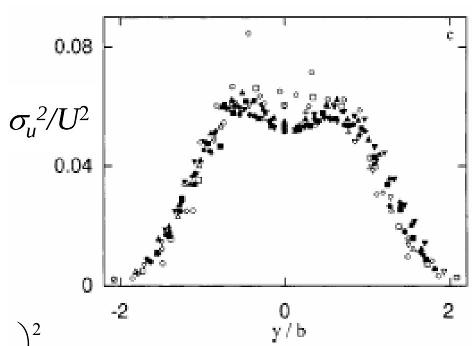
The above statement says that we want $S_{\bar{x}}$ to be less than 5% of $\bar{x}$

$$0.05 = S_{\bar{x}} / \bar{x} = \frac{S_x}{\bar{x} N^{1/2}}$$

$$N = \left(\frac{S_x}{0.05\,\bar{x}} \right)^2$$

Required Samples

But how do we know S_x?



σ_u^2/U^2

$$N = \left(\frac{\sigma_u}{X\%U}\right)^2$$

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