Uncertainty Analysis

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This document is intended to introduce the reader with a brief introduction to uncertainty analysis and the propagation of uncertainties through an algorithm to a result. Uncertainty analysis is not a topic that is generally covered in Rigid Body Dynamics courses, which is unfortunate because it is important for engineers to assess the quality of their predictions. The question that every engineer should ask him, or herself, when performing a calculation is: "Assuming my model is correct, and given the uncertainty in my measurements, what is the uncertainty in my prediction?"

Definitions:

It is important that we all get started off on the same footing. With that in mind, I've included a few "definitions".

- *Accuracy*: The deviation of a reading from a known input, generally expressed as a percentage of the full scale reading. For example, imagine a meter stick has a full scale range of 1000mm and an accuracy of ±5mm, or ±0.5%, over that range. This uncertainty in the accuracy could be related to a number of things, but one possibility is the material that the meter stick is constructed from. Although it's unlikely, a wooden may fluctuate in length by that much with changes in the humidity.

- *Precision:* The ability of an instrument to reproduce certain readings with a given accuracy. For example, a known $100V$ potential is to be measured with a particular voltmeter. The voltmeter returns: $103V$, $104V$, $105V$, $103V$, and $105V$. The accuracy of the voltmeter is roughly 5% but the precision is ±1%.

- *Least Count:* The smallest difference between the demarkations on the scale of an instrument. For example, imagine a meter stick where the smallest demarkations are spaced 1cm apart. The least count is 1cm and consequently, the resolution of this meter stick would be ±5mm, or ±1/2 of the least count.

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Uncertainty Analysis

Uncertainty analysis is the process of systematically quantifying those estimates. Whenever we make a measurement there is an associated uncertainty: $x' = \overline{x} \pm u_x(P\%)$. Uncertainty analysis provides us with the tools to estimate u_x for a desired probability level ($P\%$), i.e., there is a 95% probability that the measurement will fall within the range specified by $\overline{x}$ and $\pm u_x$.

Elemental Uncertainties

Each element of error present within a measurement will combine with other errors to increase the uncertainty of a measurement. Often, we are interested in measuring x where x is subject to k sources, or elements, of error, e_j, for $j = 1, 2, \cdots, k$. The Root-Sum-Squares (RSS) method may be used to estimate the uncertainty, u_x, in the measured quantity x.

$$u_x = \pm\sqrt{e_1^2 + e_2^2 + \cdots + e_k^2} = \pm\sqrt{\sum_{j=1}^{k} e_j^2} \qquad (1)$$

One of the common, and significant, sources of uncertainty is associated with the resolution, or least count, of an instrument. This is referred to as interpolation uncertainty. A general rule of thumb is to assign a numerical value to the interpolation uncertainty, u_o, of ±1/2 the instrument resolution at a probability of 95%, i.e., $u_o = \pm 1/2$ resolution (95%). The (95%) probability implies that there is a 20 to 1, or a 5%, chance that a value will exceed the interval u_o.

Problem: Example 1

An automobile speedometer is graduated in 5mph increments and has an accuracy rated to be within ±4%. Estimate the uncertainty in the indicated speed at 60mph.

Solution:

The uncertainty in the speed, u_s, is a function of the interpolation uncertainty in the speedometer, $u_\circ$, and the instrument uncertainty, u_c.

$$u_s = \pm\sqrt{u_\circ^2 + u_c^2} \tag{2}$$

The interpolation uncertainty, $u_\circ$, is related to the least count of the instrument. The interpolation uncertainty of the speedometer in our example is ±1/2 the least count or ±2.5mph. On the other hand, the instrument uncertainty in this problem is related to the accuracy of the speedometer, and can probably be traced back to the manufacturing tolerances used in the construction of the device.

$$u_s = \pm\sqrt{(2.5)^2 + (0.04 \times 60)^2)} \tag{3}$$

After plugging the appropriate values into equation 2 and solving, we find the uncertainty in the speed to be:

$$u_s = \pm 3.5mph \tag{4}$$

Propagation of Uncertainties

Often functional relationships are used in conjunction with measured variables to determine a variable of interest. For example, the uncertainty associated with estimates of kinetic energy, T, is a function of the density of the body, ρ, the volume of the body, ϑ, and the velocity of the body, V:

$$C_L = f(\rho, \vartheta, V) \tag{5}$$

However, the value and uncertainty of T is more sensitive to changes in some of these quantities, such as the velocity, V, than others. The remainder of this section will discuss how we account for the sensitivity of an equation to changes in different variables.

Single-Variable Problems

The true value of $\overline{y}$ depends on the sensitivity in the measurement of $\overline{x}$.

$$\overline{y} \pm \delta y = f(\overline{x} \pm tS_{\overline{x}}) \tag{6}$$

Performing a Taylor Series Expansion on equation 6, yields:

$$\overline{y} \pm \delta y = f(\overline{x}) \pm \left[\left(\frac{dy}{dx}\right)_{x=\overline{x}} tS_{\overline{x}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{x=\overline{x}} tS_{\overline{x}} + \cdots\right] \tag{7}$$

Assume that for small changes a linear approximation is valid:

$$\overline{y} \pm \delta y = f(\overline{x}) \pm \left(\frac{dy}{dx}\right)_{x=\overline{x}} tS_{\overline{x}}. \tag{8}$$

By inspection we can split this equation according to the measurement and it's associated uncertainty and we can see that in general, the uncertainty in y, u_y, is proportional to the uncertainty in x, u_x, as:

$$u_y = \left(\frac{dy}{dx}\right)_{x=\overline{x}} u_x \tag{9}$$

This concept is demostrated in Figure 1, where the uncertainty in y is clearly related to the uncertainty in x and the slope of line related x to y.

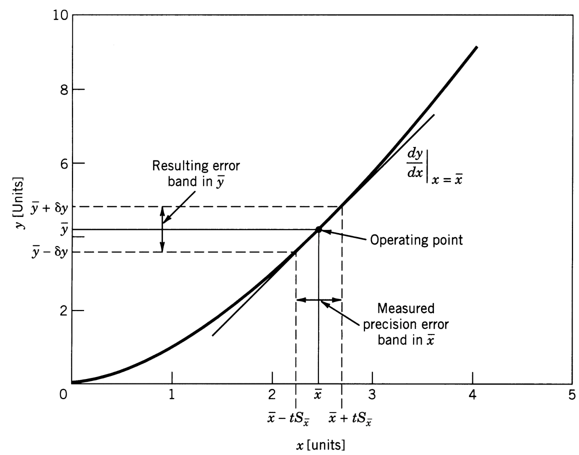


Figure 1: Relationship between the uncertainty of a measured variable, $\overline{x} \pm tS_{\overline{x}}$, and the uncertainty in a calculated quantity, $\overline{y} \pm \delta_{\overline{y}}$. – From Figliola and Beasley (2000)[1]

Multi-Variable Problems

To propagate uncertainties through equations containing multiple variables, we'll combine the RSS approach used to combine elemental uncertainties with the approach described for single variable problems in the last section. Begin by letting R be the result determined from L independent variables, x_i, such that, $R = f_1(x_1, x_2, \cdots, x_L)$. If we assign the uncertainties for each of the independent variables at the same probablility level, then the uncertainty in R, can be defined in functional form as:

$$u_r = f_2(u_{x_1} + u_{x_2} + \cdots u_{x_L}) \tag{10}$$

which then yields an equation for the multi variable problem of the form:

$$u_r = \pm\sqrt{\sum_{i=1}^{L}\left(\left.\frac{\partial R}{\partial x_i}\right|_{x=\overline{x_i}} u_{x_i}\right)^2} \tag{11}$$

To demonstrate the application of uncertainty analysis to the multi-variable problems a pair of examples are provided.

Problem: Example 2

Estimate the uncertainty in the density of a cylinderical bar given the mass, $m = 4.5 \pm 0.1$lbm, the diameter, $d = 4 \pm 0.05$in., and the length, $l = 6 \pm 0.005$in.

Solution:

The density, ρ, is defined as:

$$\rho = \frac{m}{v} \tag{12}$$

where the volume, v, is:

$$v = \frac{1}{4}\pi d^2 l \tag{13}$$

Substituting Equation 13 into Equation 12:

$$\rho = \frac{4m}{\pi d^2 l} \tag{14}$$

Applying equation 11 to equation 14:

$$u_\rho = \sqrt{\left(\frac{\partial \rho}{\partial m} u_m\right)^2 + \left(\frac{\partial \rho}{\partial d} u_d\right)^2 + \left(\frac{\partial \rho}{\partial l} u_l\right)^2} \tag{15}$$

Evaluating each of the partial derivatives:

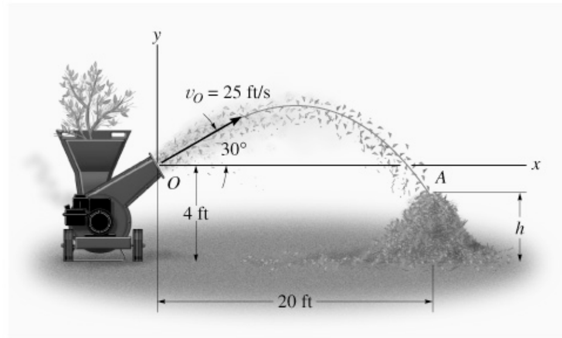
$$u_\rho = \sqrt{\left(\frac{4}{\pi d^2 l} u_m\right)^2 + \left(\frac{-8m}{\pi d^3 l} u_d\right)^2 + \left(\frac{4m}{\pi d^2 l^2} u_l\right)^2} \tag{16}$$

Inserting the measured values and their associated uncertainties into Equation 16 and evaluating Equation **??** yields:

$$\rho \pm u_\rho = 103.13 \pm 3.45 \frac{lbm}{ft^3} \tag{17}$$

Problem: Example 3[2]

The chipping machine is designed to eject wood chips at $v_o = 25 \pm 0.1 ft/s$. If the exit nozzle is square with $2in.$ on each side and oriented at $\theta = 30° \pm 2.5°$ from the horizontal, determine how high, h, the chips strike the pile if they land on the pile $20ft$ from the exit. Note: the distance from the ground to the centerline of the nozzle exit is $4ft \pm 1in.$



Solution:

We can begin by breaking the exit velocity, v_0, of the wood chips into horizontal, v_{0_x}, and vertical, v_{0_y}, components.

$$v_{0_x} = v_0 \cos(\theta) \tag{18}$$

$$v_{0_y} = v_0 \sin(\theta) \tag{19}$$

We can now write out an equation to describe the horizontal motion. Notice that, in the absence of an aerodynamic drag force, there is no acceleration in the x direction.

$$x_A = x_0 + v_{0_x} t_{0_A} \tag{20}$$

The time required for the wood chips to travel the 20ft can be found by rearranging equation 20 as follows:

$$t_{0_A} = \frac{x_A - x_0}{v_{0_x}}. \tag{21}$$

To describe the vertical motion of the wood chips, we need to include an acceleration term. The acceleration in this case is assumed to be $a_c = -g = -32.2 ft/s$.

$$h = y_A = y_0 + v_{0_y} t_{0_A} + \frac{1}{2} a_c t_{0_A}^2 \tag{22}$$

We can now substitute equations 18, 19, and 21 into equation 23 to obtain an equation that relates the height of the chip pile to the exit conditions of the chipper.

$$h = y_0 + (x_A - x_0)\frac{\sin(\theta)}{\cos(\theta)} + \frac{a_c}{2}\left[\frac{x_A - x_0}{v_0 \cos(\theta)}\right]^2 \tag{23}$$

Before continuing, let's define a new variable, s, where $(x_A - x_0) = s = 20ft \pm 1/2in.$. Without rewriting, equation 23, we can see that the height, h is a function of the nozzle exit height, the horizontal displacement of the chips, the exit angle, and the exit velocity of the chips: $h(y_0, s, \theta, v_0)$. At this point, we can "plug-in" the numbers that we were supplied with and compute h, but we're also interested in knowing how good our estimate of h is. To determine this, we'll begin by computing the partial derivative of h with respect to each of the variables with which h depends upon. Equations 24 through 27 present the partial derivatives of h with respect to y_0, s, θ, and v_0.

$$\frac{\partial h}{\partial y_0} = 1 \tag{24}$$

$$\frac{\partial h}{\partial s} = \frac{\sin(\theta)}{\cos(\theta)} + \frac{a_c s}{v_0^2 \cos(\theta)^2} \tag{25}$$

$$\frac{\partial h}{\partial \theta} = s + \frac{s \sin(\theta)^2}{\cos(\theta)^2} + \frac{a_c s^2 \sin(\theta)}{v_0^2 \cos(\theta)^3} \tag{26}$$

$$\frac{\partial h}{\partial v_0} = \frac{a_c s^2}{v_0^3 \cos(\theta)^2} \tag{27}$$

With the partial derivatives computed, we can now form an equation to evaluate the uncertainty, u_h, in our estimate of h. This is accomplished within a root-sum-of-squares equation where each term of the summation is formed by combining the uncertainty in each of the variables with the associated partial derivative of that variable as shown in equation 28.

$$u_h = \pm\Big[\Big(\frac{\partial h}{\partial y_0} u_{y_o}\Big)^2 + \Big(\frac{\partial h}{\partial s} u_s\Big)^2 + \Big(\frac{\partial h}{\partial \theta} u_\theta\Big)^2 + \Big(\frac{\partial h}{\partial v_0} u_{v_o}\Big)^2\Big]^{\frac{1}{2}} \tag{28}$$

Evaluating equations 23 and 29 we find the height of the pile to be:

$$h = 1.81 \pm 0.49 ft = 21.7 \pm 5.9 in. \tag{29}$$

The important thing to notice is that performing the uncertainty analysis does not alter the estimate of h. It does, however, allow us to evaluate how small uncertainties in the variables may impact our ability to estimate the motion of a body. In this case, the uncertainties assigned to each variable combine to produce an uncertainty that's ±27% of our predicted height.

References:

1. Figliola, R.S., and Beasley, D.E., (2000) *Theory and Design for Mechanical Measurements, 3rd Edition*, John Wiley & Sons, Inc., New York, New York, USA, Chapter 4 – 5
2. Hibbeler, R.C., (2004) *Engineering Mechanics: Dynamics, 10th Edition*, Pearson Prentice-Hall, Upper Saddle River, New Jersey, USA, pp. 41