Solving $F(z+1) = b^{F(z)}$ in the complex plane

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Abstract

The generalized tetration, defined by the equation $F(z+1) = b^{F(z)}$ in the complex plane with $F(0) = 1$, is considered for any $b > e^{1/e}$. By comparing other solutions to Kneser's solution, natural conditions are found which force Kneser's solution to be the unique solution to the equation. This answers a conjecture posed by Trappmann and Kouznetsov. Also, a new iteration method is developed which numerically approximates the function $F(z)$ with an error of less than 10^{-50} for most bases b, using only 180 nodes, with each iteration gaining one or two places of accuracy. This method can be applied to other problems involving the Abel equation.

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1 Background

Throughout history, mathematicians have faced the problem of representing larger and larger numbers. Exponential notation was a huge improvement over standard representations; however, there still can be a problem with "floating overflow" if the exponent gets too large. One solution is to find a function with super-fast growth [16]. One proposed function would be the *tetration* function, whose name comes from *tetra-* (four) and *iteration*. Addition by a positive integer n can be thought of as repeated incrementation, multiplication by n is done by repeated addition, and exponentiation by n is repeated multiplication. So a fourth level operation would be repeated exponentiation using the initial base b:

$$\underbrace{b^{b^{\cdot^{\cdot^{b}}}}}_{n}$$

Although there are many different notations for this operation, we will use the iterated exponential notation

$$\exp_b^n(x) = b^{b^{\cdot^{\cdot^{b^x}}}} \qquad \text{with } n \text{ } b\text{'s.}$$

The tetration function for base b can then be expressed as $F(n) = \exp_b^n(1)$. Thus, we see that $F(0) = 1$, $F(1) = b$, $F(2) = b^b$, and so on. In fact, we can recursively define $F(x)$ by $F(x+1) = b^{F(x)}$. Working backwards from this relation, we see that $F(-1) = 0$ and $F(-2)$ must be undefined. The question is whether we can extend $F(x)$ to the whole complex plane, minus a branch cut at $x \le -2$.

If we have a tetration function $F(x)$, the inverse function $F^{-1}(x)$ solves Abel's functional equation [1]

$$\alpha(g(x)) = \alpha(x) + 1$$

for the function $g(x) = b^x$. Such a function can be referred to as an *arctetration*, which we will denote by $\alpha(z)$. An analytic solution to Abel's equation allows us to find fractional iterations of the function $g(x)$. For example, if we let $f(x) = \alpha^{-1}(\alpha(x) + 1/2)$, then $f(f(x)) = g(x)$, and so we have found a "half-iterate" of $g(x)$. In fact, solving for $\alpha(x)$ allows us to solve a variety of functional equations involving $g(x)$ [11].

The problem, though, is that the solution to Abel's equation is far from unique. Not only can we add an arbitrary constant, but if $p(x)$ is any periodic function of period 1 such that $p'(x) > -1$, then $\alpha(x) + p(\alpha(x))$ will also be a solution to Abel's equation. In order to establish uniqueness for functions with no real fixed points, as is the case for $g(x) = b^x$, we need to extend our focus to the complex plane.

It should be noted that if $e^{-e} \le b \le e^{1/e}$, then there is a real fixed point of b^x, and therefore a unique real analytic solution can be found to the tetration problem using standard power series techniques. The special case $b = \sqrt{2}$ was analyzed in [9] and [10]. The case $b = e^{1/e}$ is covered extensively in [15] and [17]. Hence, this paper will only be concerned with the case $b > e^{1/e} \approx 1.44466786$.

In [14], criterion were found for which there was a unique holomorphic solution $\alpha(z)$ to Abel's equation for the case $b > e^{1/e}$. In fact, they show that Kneser's solution from [5] satisfies these criteria. Unfortunately, [14] only conjectures that there is a unique holomorphic tetration function $F(z)$ satisfying the condition $F(x+1) = b^{F(z)}$. In [7], a method was presented for computing the tetration function $F(z)$ to about 14 decimal places, at least for the base $b = e$. However, this precision could only be obtained by using over 2000 nodes for a Gaussian quadrature, which slowed down the computation time.

In this paper, we will find a method for computing $F(z)$ to almost 50 places for all bases $b > e^{1/e}$. This method converges much faster in terms of the number of calculations than [7], since only 180 nodes are needed to obtain this accuracy for $b = e$. In the process, we will be able the prove the uniqueness conjecture proposed by [7].

2 Kneser's solution

We will begin by reproducing Kneser's result from [5]. Actually, Kneser only considered the function e^z, so we will generalize his result to include b^z. We also will reformat his solution to make it easier to refer to different aspects of his solution later on.

We begin by solving the Schröder equation $\sigma_b(b^z) = s\sigma_b(z)$, where s is the derivative of b^z at a fixed point [13]. We are only interested in the case $b > e^{1/e}$, for which there are only complex fixed points. Then we let L_b be the complex solution of the equation $b^z = z$ for which $\Im(L_b) > 0$, with $\Im(L_b)$ minimized. Then $s = \ln(L_b) = L_b \ln(b)$. There is a unique solution to Schröder's equation which is analytic at L_b, and for which $\sigma_b'(L_b) = 1$ [6]. By equating coefficients of the power series, we find that

$$\begin{aligned}
\sigma_b(z) &= (z - L_b) + \frac{s}{2L_b(1-s)}(z-L_b)^2 + \frac{s^2(1+2s)}{6L_b^2(1-s)(1-s^2)}(z-L_b)^3 \\
&+ \frac{s^3(1+6s+5s^2+6s^3)}{24L_b^3(1-s)(1-s^2)(1-s^3)}(z-L_b)^4 + \frac{s^4(1+14s+24s^2+45s^3+46s^4+26s^5+24s^6)}{120L_b^4(1-s)(1-s^2)(1-s^3)(1-s^4)}(z-L_b)^5 \\
&+ \frac{s^5(1+30s+89s^2+214s^3+374s^4+416s^5+511s^6+461s^7+330s^8+154s^9+120s^{10})}{720L_b^5(1-s)(1-s^2)(1-s^3)(1-s^4)(1-s^5)}(z-L_b)^6 \\
&+ \cdots.
\end{aligned}$$

Unfortunately, it is unclear what the radius of convergence will be for this series. However, we can easily extend the region of convergence to cover the upper half plane, since this is in the basin of attraction for the

fixed point L_b of the function $\log_b(z)$. That is, if we continually apply the principal value of $\log_b$ to a point in the upper half plane, it will eventually converge to L_b. Hence, we can define σ_b another way:

$$\sigma_b(z) = \lim_{n\to\infty} s^n(\log_b^n(z) - L_b),$$

where $\log_b^n(z)$ means we apply the principal branch of $\log_b(z)$ a total of n times. In fact, this representation of $\sigma_b(z)$ also works on the real axis, except for $z = 0, 1, b, b^b, b^{b^b}, b^{b^{b^b}}, \ldots$.

Next, we let $\psi_b(z) = \ln(\sigma_b(z))/\ln(s)$, so that $\psi_b(z)$ will solve Abel's equation $\psi_b(b^z) = \psi_b(z) + 1$. We can compute

$$\begin{aligned}
\psi_b(z) &= \frac{\ln(z-L_b)}{\ln(s)} + \frac{s}{2L_b\ln(s)(s-1)}(z-L_b) + \frac{s^2(1+5s)}{24L_b^2\ln(s)(1-s)(1-s^2)}(z-L_b)^2 \\
&+ \frac{s^4(2+s+3s^2)}{24L_b^3\ln(s)(1-s)(1-s^2)(1-s^3)}(z-L_b)^3 \\
&+ \frac{s^4(61s-1+71s^2+290s^3+299s^4+109s^5+251s^6)}{2880L_b^4\ln(s)(1-s)(1-s^2)(1-s^3)(1-s^4)}(z-L_b)^4 \\
&+ \frac{s^6(6+15s+69s^2+143s^3+115s^4+212s^5+221s^6+155s^7+49s^8+95s^9)}{1440L_b^5\ln(s)(1-s)(1-s^2)(1-s^3)(1-s^4)(1-s^5)}(z-L_b)^5 \\
&+ \cdots.
\end{aligned}$$

We can compute $\psi_b(z)$ very quickly to high levels of precision by the formula

$$\psi_b(z) = \lim_{n\to\infty} \frac{\ln(s^n T_m(\log_b^n(z)))}{\ln(s)},$$

where $T_m(z)$ is the m-th degree Taylor polynomial for $\sigma_b(z)$. Increasing m will increase the rate of convergence.

Next, we find the inverse function $\psi_b^{-1}(z) = \sigma_b^{-1}(e^{z\ln s})$. This function will satisfy the functional equation

$$\psi_b^{-1}(z+1) = b^{\psi_b^{-1}(z)}.$$

By reversing the series for $\sigma_b(z)$, we find that

$$\begin{aligned}
\psi_b^{-1}(z) &= L_b + e^{z\ln s} - \frac{s}{2L_b(1-s)}e^{2z\ln s} + \frac{s^2(2+s)}{6L_b^2(1-s)(1-s^2)}e^{3z\ln s} \\
&- \frac{s^3(6+6s+5s^2+s^3)}{24L_b^3(1-s)(1-s^2)(1-s^3)}e^{4z\ln s} + \frac{s^4(24+36s+46s^2+40s^3+24s^4+9s^5+s^6)}{120L_b^4(1-s)(1-s^2)(1-s^3)(1-s^4)}e^{5z\ln s} \\
&- \frac{s^5(120+240s+390s^2+480s^3+514s^4+416s^5+301s^6+160s^7+64s^8+14s^9+s^{10})}{720L_b^5(1-s)(1-s^2)(1-s^3)(1-s^4)(1-s^5)}e^{6z\ln s} \\
&+ \cdots.
\end{aligned}$$

As long as $b > e^{1/e}$, then $\Re(\ln(s)) > 0$, so we can express $\psi_b^{-1}(z)$ as a limit.

$$\psi_b^{-1}(z) = \lim_{n\to\infty} \exp_b^n(L_b + e^{(z-n)\ln s}).$$

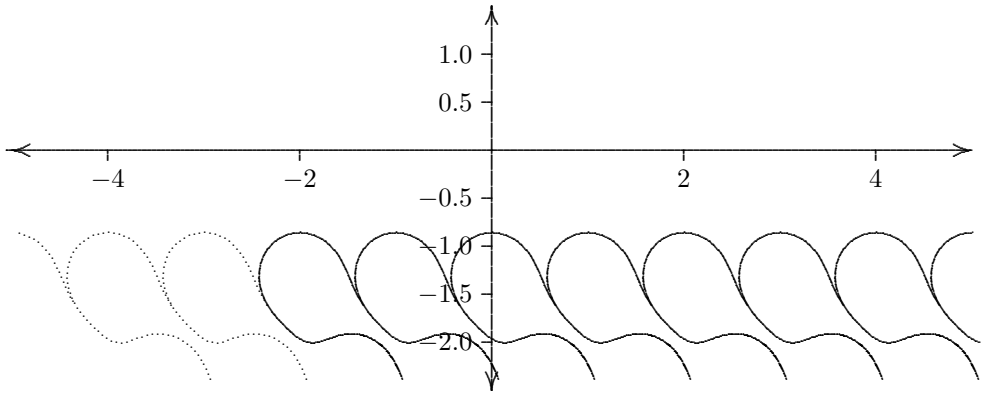

Figure 1: The set T_b is shown in the solid lines for $b = e$. The set T_b' also includes the dotted lines.

In fact, we can speed up the convergence by considering the limit

$$\psi_b^{-1}(z) = \lim_{n\to\infty} \exp_b^n(S_m(e^{(z-n)\ln s})), \tag{1}$$

where $S_m(z)$ is the m-th degree Taylor polynomial for $\sigma_b^{-1}(z)$. We can use this limit to show that $\psi_b^{-1}(z)$ is an entire function for $b > e^{1/e}$.

Since $\psi_b(z)$ is defined for real numbers except $z = 0, 1, b, b^b, b^{b^b}, \ldots$, we can let T_b be the set $\{\psi_b(x) \mid x \in \mathbb{R} - \{0, 1, b, b^b, \ldots\}\}$. Note that if $z \in T_b$, then $z + 1 \in T_b$ as well, since $\psi_b(b^x) = \psi_b(x) + 1$. If we let T_b' be the set

$$T_b' = \{z - k \mid z \in T_b \text{ and } k \in \mathbb{Z}^+\},$$

then T_b' will be an extension of the set T_b, so that the pattern extends to the left as well: $z-1 \in T_b'$ whenever $z \in T_b'$. Figure 1 shows these two sets for the case $b = e$.

Let R_b be the simply connected open set above T_b'. Let z_0 be any point in R_b. By the Riemann mapping theorem, there is a bijective holomorphic mapping from R_b to the upper half plane. By following this mapping with a Möbius transformation $z \mapsto (az+b)/(cz+d)$, $ad-bc > 0$, we can get a holomorphic mapping ρ_b from R_b to the upper half plane such that $\rho_b(z_0 + 1) = \rho_b(z_0) + 1$ and $\rho_b'(z_0 + 1) = \rho_b'(z_0)$. Since the region R_b is periodic, we can define the mapping $\rho_b(\rho_b^{-1}(z)+1)-1$ sending the upper half plane to the upper half plane, which will fixes the point $\rho(z_0)$, and whose derivative at $\rho(z_0)$ is 1. Thus, $\rho_b(\rho_b^{-1}(z) + 1) - 1 = z$ for all z, allowing us to establish the identity $\rho_b(z + 1) = \rho_b(z) + 1$ for all $z \in R_b$. Likewise, $\rho_b^{-1}(z + 1) = \rho_b^{-1}(z) + 1$ for all z in the upper half plane.

Even though $\rho_b(z)$ is only defined for the region R_b, since the boundary T_b is piecewise smooth, $\rho_b(z)$ can be extended continuously to the set T_b'. Since we can add a real constant to $\rho_b(z)$ to produce a function with the same properties, we will add such a constant so that

$$\lim_{x\to-\infty} \rho_b(\psi_b(x)) = -2.$$

Finally, we consider the function $\kappa_b(z) = \psi_b^{-1}(\rho_b^{-1}(z))$. This will be defined for the upper half-plane, and in this region

$$\kappa_b(z + 1) = \psi_b^{-1}(\rho_b^{-1}(z) + 1) = b^{\psi_b^{-1}(\rho_b^{-1}(z))} = b^{\kappa_b(z)}.$$

Furthermore, for $x > -2$, $x \notin \mathbb{Z}$, $\kappa_b(x)$ is defined, and in fact is real. Since $\lim_{x\to -2^+} \kappa_b(x) = -\infty$, we see that $\lim_{x\to -1^+} \kappa_b(x) = 0$, $\lim_{x\to 0^+} \kappa_b(x) = 1$, $\lim_{x\to 1^+} \kappa_b(x) = b$, etc., so we can extend the definition of $\kappa_b(x)$ by continuity to all $x > -2$. Finally, since $\kappa_b(z)$ sends real numbers to real numbers, we can analytically extend $\kappa_b(z)$ through complex conjugation $\kappa_b(\overline{z}) = \overline{\kappa_b(z)}$ to the cut plane

$$C_{-2} = \mathbb{C} - \{x \in \mathbb{R} \mid x \leq -2\}.$$

Then everywhere in this cut plane, we have $\kappa_b(z+1) = b^{\kappa_b(z)}$.

3 The Uniqueness of Kneser's solution

Both [7] and [14] conjecture that there is a unique analytical function $F(z)$ that satisfies $F(z+1) = b^{F(z)}$ on C_{-2} for which $F(\overline{z}) = \overline{F(z)}$. Clearly Kneser's solution $\kappa_b(z)$ satisfies these conditions, but the question is whether any other functions could also satisfy the same conditions.

To answer this question, we need to understand the properties of the $\rho_b(z)$ function used in the construction of Kneser's solution. Since $\rho_b(z+1) = \rho_b(z) + 1$, the function $\rho_b(z) - z$ will be a periodic function of period 1. Thus, there is a complex Fourier series for $\rho_b(z) - z$:

$$\rho_b(z) - z = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k z}.$$

But $\rho_b(z)$ is also a conformal mapping from the region R_b to the half plane, and so it solves a fluid flow problem on the region R_b. Although we can expect major changes in the fluid flow near the boundary T_b', we see that the fluid flow will approximate a linear flow far away from the boundary. Thus, $\rho_b(z) - z$ will approach a constant as $\Im(z) \to \infty$. In order for this to happen in the Fourier series, we see that $c_k = 0$ for all $k < 0$. Thus,

$$\rho_b(z) = z + \sum_{k=0}^{\infty} c_k e^{2\pi i k z} = z + c_0 + c_1 e^{2\pi i z} + c_2 e^{4\pi i z} + c_3 e^{6\pi i z} + \cdots.$$

It is possible to reverse this series, and find a series for $\rho_b^{-1}(z)$:

$$\rho_b^{-1}(z) = z + d_0 + d_1 e^{2\pi i z} + d_2 e^{4\pi i z} + d_3 e^{6\pi i z} + \cdots, \tag{2}$$

where

$$\begin{aligned}
d_0 &= -c_0 \\
d_1 &= -c_1 e^{-2\pi i c_0} \\
d_2 &= (-c_2 + 2\pi i c_1^2) e^{-4\pi i c_0} \\
d_3 &= (-c_3 + 6\pi i c_1 c_2 + 6\pi^2 c_1^3) e^{-6\pi i c_0} \\
d_4 &= \left(-c_4 + 8\pi i c_1 c_3 + 4\pi i c_2^2 + 32\pi^2 c_1^2 c_2 - \frac{64}{3}\pi^3 i c_1^4 \right) e^{-8\pi i c_0} \\
d_5 &= \left(-c_5 + 10\pi i c_2 c_3 + 10\pi i c_1 c_4 + 50\pi^2 c_1 c_2^2 + 50\pi^2 c_1^2 c_3 - \frac{500}{3}\pi^3 i c_1^3 c_2 - \frac{250}{3}\pi^4 c_1^5 \right) e^{-10\pi i c_0}
\end{aligned}$$

$$
\begin{aligned}
d_6 &= \Big(- c_6 + 6\pi i c_3^2 + 12\pi i c_2 c_4 + 12\pi i c_1 c_5 + 24\pi^2 c_2^3 + 144\pi^2 c_1 c_2 c_3 + 72\pi^2 c_1^2 c_4 - 432\pi^3 i c_1^2 c_2^2 \\
&\quad - 288\pi^3 i c_1^3 c_3 - 864\pi^4 c_1^4 c_2 + \frac{1728}{5}\pi^5 i c_1^6 \Big) e^{-12\pi i c_0} \\
d_7 &= \Big(- c_7 + 14\pi i c_3 c_4 + 14\pi i c_2 c_5 + 14\pi i c_1 c_6 + 98\pi^2 c_2^2 c_3 + 98\pi^2 c_1 c_3^2 + 196\pi^2 c_1 c_2 c_4 + 98\pi^2 c_1^2 c_5 \\
&\quad - \frac{1372}{3}\pi^3 i c_1 c_2^3 - 1372\pi^3 i c_1^2 c_2 c_3 - \frac{1372}{3}\pi^3 i c_1^3 c_4 - \frac{9604}{3}\pi^4 c_1^3 c_2^2 - \frac{4802}{3}\pi^4 c_1^4 c_3 + \frac{67228}{15}\pi^5 i c_1^5 c_2 \\
&\quad + \frac{67228}{45}\pi^6 c_1^7 \Big) e^{-14\pi i c_0} \\
d_8 &= \Big(- c_8 + 8\pi i c_4^2 + 16\pi i c_3 c_5 + 16\pi i c_2 c_6 + 16\pi i c_1 c_7 + 128\pi^2 c_2 c_3^2 + 128\pi^2 c_2^2 c_4 + 256\pi^2 c_1 c_3 c_4 \\
&\quad + 256\pi^2 c_1 c_2 c_5 + 128\pi^2 c_1^2 c_6 - \frac{512}{3}\pi^3 i c_2^4 - 2048\pi^3 i c_1 c_2^2 c_3 - 1024\pi^3 i c_1^2 c_3^2 - 2048\pi^3 i c_1^2 c_2 c_4 \\
&\quad - \frac{2048}{3}\pi^3 i c_1^3 c_5 - \frac{16384}{3}\pi^4 c_1^2 c_2^3 - \frac{32768}{3}\pi^4 c_1^3 c_2 c_3 - \frac{8192}{3}\pi^4 c_1^4 c_4 + \frac{65536}{3}\pi^5 i c_1^4 c_2^2 \\
&\quad + \frac{131072}{15}\pi^5 i c_1^5 c_3 + \frac{1048576}{45}\pi^6 c_1^6 c_2 - \frac{2097152}{315}\pi^7 i c_1^8 \Big) e^{-16\pi i c_0}
\end{aligned}
$$

Because the function $\rho_b^{-1}(z)$ is analytic in the upper half plane, the auxiliary function

$$g_b(z) = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots \tag{3}$$

will have a radius of convergence of 1. Then $\rho_b^{-1}(z) = z + g_b(e^{2\pi i z})$.

Proposition 1:

Suppose that $F(z)$ is an analytic function defined on C_{-2} for which $F(z+1) = b^{F(z)}$, $F(0) = 1$, and that $F(\overline{z}) = \overline{F(z)}$. Suppose further that for the region where $\Im(F(z)) > 0$, $\psi_b(F(z))$ can be expressed as

$$\psi_b(F(z)) = z + \sum_{k=0}^{\infty} f_k e^{2\pi i k z}.$$

Then $F(z) = \kappa_b(z)$ for all $z \in C_{-2}$.

Proof:

Note that

$$\kappa_b^{-1}(F(z+1)) = \kappa_b^{-1}(b^{F(z)}) = \kappa_b^{-1}(F(z)) + 1.$$

Thus, $\kappa_b^{-1}(F(z)) - z$ is periodic with period 1. Since $\kappa_b^{-1}(z) = \rho_b(\psi_b(z))$, we find that

$$\kappa_b^{-1}(F(z)) = \rho_b\left(z + \sum_{k=0}^{\infty} f_k e^{2\pi i k z}\right) = z + \sum_{k=0}^{\infty} h_k e^{2\pi i k z},$$

where we can determine h_k formally from the c_k and f_k. Since $F(\overline{z}) = \overline{F(z)}$,

$$\kappa_b^{-1}(F(z)) = \overline{\kappa_b^{-1}(F(\overline{z}))} = z + \sum_{k=0}^{\infty} h_k e^{-2\pi i k z}.$$

By the uniqueness of the Fourier coefficients, this forces $h_k = 0$ for all $k \neq 0$. Thus, $\kappa_b^{-1}(F(z)) = z + h_0$. But $\kappa_b^{-1}(F(0)) = \kappa_b^{-1}(1) = 0$, so $h_0 = 0$. Since $\kappa_b^{-1}(F(z)) = z$ whenever $\Im(F(z)) > 0$, we see that $F(z) = \kappa_b(z)$ for at least some region, and by analytic continuation, $F(z) = \kappa_b(z)$ for all z in C_{-2}.

□

Proposition 2:

Suppose that $F(z)$ is an analytic function defined on C_{-2} for which $F(z+1) = b^{F(z)}$, $F(0) = 1$, and that $F(\overline{z}) = \overline{F(z)}$. Suppose that for all $x \in \mathbb{R}$,

$$\lim_{y \to +\infty} F(x + iy) = L_b.$$

Then $F(z) = \kappa_b(z)$ for all $z \in C_{-2}$.

Proof:

Since $\psi_b(F(z+1)) = \psi_b(b^{F(z)}) = \psi_b(F(z)) + 1$, we see that $\psi_b(F(z)) - z$ is periodic with period 1. Since $\Im(L_b) > 0$, by the limit property, we have that $\Im(F(z)) > 0$ whenever $0 \leq \Re(z) \leq 1$ and $\Im(z) > M$ for sufficiently large M, and by periodicity, we can define $\psi_b(F(z))$ whenever $\Im(z) > M$. By the Fourier series,

$$\psi_b(F(z)) = z + \sum_{k=-\infty}^{\infty} f_k e^{2\pi i k z}.$$

Thus,

$$F(z) = \psi_b^{-1}\left(z + \sum_{k=-\infty}^{\infty} f_k e^{2\pi i k z}\right).$$

If $f_k \neq 0$ for some negative k, then as $\Im(z) \to \infty$,

$$\left|f_k e^{2\pi i k z}\right| \to \infty \quad \text{as} \quad \Im(z) \to \infty.$$

If there are many such terms, they will have different exponential growth rates asymptotically, so such terms cannot cancel out asymptotically. Thus

$$\left|\sum_{k=-\infty}^{\infty} f_k e^{2\pi i k z}\right| \to \infty \quad \text{as} \quad \Im(z) \to \infty.$$

Also note that the argument of this sum will depend on the real part of z. Thus, there is a path C going towards $+\infty i$ for which $\psi_b(F(z))$ is real and positive along this path. That is, for every $y > 0$ there is an x such that with $z = x + iy$,

$$\arg\left(z + \sum_{k=-\infty}^{\infty} f_k e^{2\pi i k z}\right) = 0.$$

But then along this path,

$$\lim_{C} F(z) = \lim_{C} \psi_b^{-1}\left(z + \sum_{k=-\infty}^{\infty} f_k e^{2\pi i k z}\right) = \lim_{z \to +\infty} \psi_b^{-1}(z),$$

and $\lim_{z \to +\infty} \psi_b^{-1}(z)$ does not exist, since the real part is unbounded, and the imaginary part is negative. (ψ_b^{-1} maps the real axis to the set T_b.) This contradicts the fact that $\Im(F(z)) > 0$ whenever $\Im(z) > M$. This contradiction shows that $f_k = 0$ for negative k. So by Proposition 1, $F(z) = \kappa_b(z)$.

□

Corollary 1:

Let $z_0 \in C_{-2}$. Then $\kappa_b(z_0)$ is a holomorphic function of the base b for $b > e^{1/e}$.

Proof:

First consider the case $\Im(z_0) > 0$. Because of uniqueness, each of the Fourier coefficients d_i for $i \geq 1$ will be a holomorphic function of b. Unfortunately, d_0 is not uniquely defined because of the branch cut of $\psi_b(z)$. Adding $2k\pi i/\ln(s)$ to d_0, where k is an integer, produces another valid d_0. But by careful manipulation of the branch cut of $\psi_b(z)$, we can make d_0 a holomorphic function of b.

Because $g_b(z)$ has a radius of convergence of 1, the series for $g_b(e^{2\pi i z_0})$ will be absolutely convergent for each b, so $g_b(e^{2\pi i z_0})$ will be a holomorphic function of b. Hence, $\rho_b^{-1}(z_0)$ is holomorphic. It is already established [4] that $\psi_b^{-1}(z)$ is a holomorphic function of b for $b > e^{1/e}$, so $\kappa_b(z_0) = \psi_b^{-1}(\rho_b^{-1}(z_0))$ is holomorphic.

Complex congugation handles the case with $\Im(z_0) < 0$. For z_0 real, $z_0 > -2$, we can consider the limit of $\kappa_b(z_0 + i\epsilon)$ as $\epsilon \to 0^+$. Since this converges uniformly to $\kappa_b(z_0)$, $\kappa_b(z_0)$ is holomorphic.

□

It is likely that $\kappa_b(z_0)$ has a logarithmic branch cut at $b = e^{1/e}$, since the tetration is not uniquely defined for $b < e^{1/e}$. [10]

4 First approximation

The goal of this paper is to produce an iterative method for calculating $\kappa_b(z)$. However, this iterative method requires having a first order approximation to begin to process. In [7], the first order approximation was a piecewise defined function which works for $b = e$, but we want to generalize the process to arbitrary b. We can do this with a polynomial approximation.

Rather than using Abel's equation $\alpha(f(z)) = \alpha(z) + 1$, we can create a homogeneous equation by letting $\lambda(z) = 1/\alpha'(z)$. Then $\lambda(z)$ satisfies *Julia's equation* [4]

$$\lambda(f(z)) = \lambda(z) f'(z).$$

In the case where $f(z) = b^z$, this becomes $\lambda_b(b^z) = \lambda_b(z) b^z \ln(b)$. One advantage of using Julia's equation over Abel's equation is that $\lambda_b(z)$ approaches 0 near the fixed point L_b, although the solution we are looking for is not analytic there. Note that Abel's solution goes to infinity as $z \to L_b$.

If we force the Maclaurin series

$$\lambda_b(z) = \sum_{n=0}^{\infty} c_n z^n$$

to satisfy $\lambda(b^z) = \lambda(z) b^z \ln b$, we find that, if we assume $c_0 = 1$, the other coefficients satisfy

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & 5 & \cdots \\ -1 & 4 - \frac{2}{\ln b} & 9 & 16 & 25 & \cdots \\ -2 & 8 - \frac{6}{\ln b} & 27 - \frac{6}{(\ln b)^2} & 64 & 125 & \cdots \\ -3 & 16 - \frac{12}{\ln b} & 81 - \frac{24}{(\ln b)^2} & 256 - \frac{24}{(\ln b)^3} & 625 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ln b - 1 \\ \ln b \\ \ln b \\ \ln b \\ \ln b \\ \vdots \end{pmatrix}, \tag{4}$$

where the pattern for the matrix is $m_{i,j} = j^{i-1} - (\ln b)^{1-j}(i-1)!/(i-j-1)!$. For each n we can solve n equations with n unknowns to find an n^{th} degree polynomial approximation for $\lambda_b(z)$. Because the matrix involves only real numbers, the polynomial approximates will be real whenever z is real.

Base b	$p_b = \lambda_b(0)$	Number of digits of precision				
		$d = 100$	$d = 200$	$d = 300$	$d = 400$	$d = 500$
3/2	1.599261338397936	31	41	44	47	49
2	1.283082409572121	17	20	22	23	24
e	1.091767351258321	17	17	18	18	19
10	0.731477430771517	11	12	13	14	15

Table 1: This shows how the polynomial approximates of $\lambda_b(z)$ seem to converge. Since p_b is calculated by an integral from 0 to 1, this gives a good indicator of the precision within this interval. Notice that the rate of convergence slows down considerably as base b increases.

The only drawback to using the homogeneous Julia's equation is that we must *normalize* the solution by multiplying by a constant p_b so that

$$\int_0^1 1/\lambda_b(z)\,dz = 1.$$

This is easily done numerically for each polynomial approximates, allowing us to approximate the solution of Abel's equation with $\alpha_b(x) = \int 1/\lambda_b(z)\,dz + C$. It is an open problem as to whether the normalized polynomial approximates converge to a function, although table 1 seems to indicate that indeed there is convergence as the degree of the polynomial increases. If it does converge, and we can pick the constant C so that $\alpha_b(1) = 0$, then by [14] this solution will be $\kappa_b^{-1}(z)$.

Even though we cannot prove convergence, for many values of b we can numerically see that it seems to slowly converge to a single function. For example, when $b = e$, we get

$$\begin{aligned}\lambda_e(z) \approx\ & 1.091767351258320991801 - 0.594439394732060169698z + 0.718664427208033464198z^2 \\ & - 0.158494299757674664300z^3 + 0.042841152742571103199z^4 - 0.010127555170266799426z^5 \\ & + 0.001690267759830729255z^6 - 0.000135002549849774665z^7 + 0.000009493791404119462z^8 \\ & - 0.000012519767005748209z^9 + 0.000002459017138186054z^{10} + 0.000001451792126232173z^{11} \\ & - 0.000000278038029203532z^{12} - 0.000000281449883192043z^{13} + 0.000000031142865920363z^{14} + \cdots.\end{aligned}$$

5 The Cross-Track Method

In this section we will show how given one approximation to the Kneser solution $\kappa_b(z)$, we can use a combination of the Fourier series and Cauchy's contour integral formula to find an even better approximation. By iterating this process, we can numerically evaluate Kneser's solution to remarkable precision.

We will use a much smaller contour than what was used in [7], which will increase the speed of the computation. Since $\kappa_b(z)$ is analytic, we have the Cauchy contour integral

$$\kappa_b(z) = \frac{1}{2\pi i}\oint_\Omega \frac{\kappa_b(t)}{t - z}\,dt,$$

where the contour Ω consists of 4 parts, where $t = x + iy$:

A: integrate along the line $x = 1$ from $t = 1 - i$ to $t = 1 + i$.

B: integrate along the upper half of the circle $x^2 + (y - 1)^2 = 1$ counterclockwise.

C: integrate along the line $x = -1$ from $t = -1 + i$ to $t = -1 - i$.

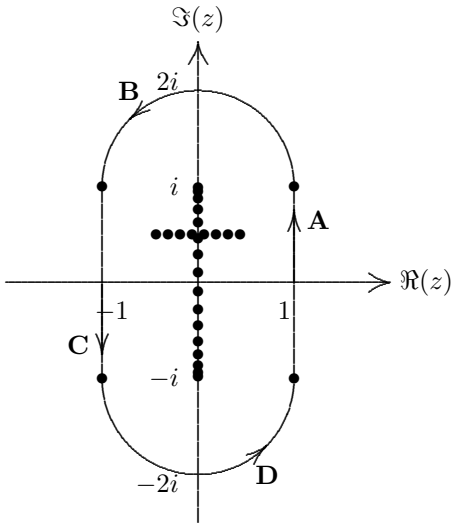

Figure 2: The contour Ω along with the interior sample points that will be used to numerically compute the contour integral. The points along the horizontal part of the cross will later be used to compute the Fourier coefficients.

D: integrate along the lower half of the circle $x^2 + (y+1)^2 = 1$ counterclockwise.
The contour Ω is shown in figure 2, and resembles a racetrack.

Note that along the integral **A**, we can let $t = 1 + iy$, and use the fact that $\kappa_b(z+1) = b^{\kappa_b(z)}$ to simplify. Thus, we see that

$$\frac{1}{2\pi i} \oint_{\mathbf{A}} \frac{\kappa_b(t)}{t - z}\, dt = \frac{1}{2\pi} \int_{-1}^{1} \frac{b^{\kappa_b(iy)}}{1 + iy - z}\, dy.$$

Likewise, integrating along **C** can be done by letting $t = -1 + iy$, along with the identity $\kappa_b(z-1) = \log_b(\kappa_b(z))$.

$$\frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{\kappa_b(t)}{t - z}\, dt = -\frac{1}{2\pi} \int_{-1}^{1} \frac{\log_b(\kappa_b(iy))}{-1 + iy - z}\, dy.$$

We can put these two pieces together to form the function

$$H(z) = \frac{1}{2\pi} \int_{-1}^{1} \left(\frac{b^{\kappa_b(iy)}}{1 + iy - z} - \frac{\log_b(\kappa_b(iy))}{-1 + iy - z} \right) dy.$$

Integrating along the curve **B** is more tricky. We can use the substitution $t = i + e^{i\theta}$ to produce

$$G(z) = \frac{1}{2\pi i} \oint_{\mathbf{B}} \frac{\kappa_b(t)}{t - z}\, dt = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\kappa_b(i + e^{i\theta}) e^{i\theta}}{i + e^{i\theta} - z}\, d\theta.$$

We can use symmetry to do the last piece of the contour. It is clear that

$$\oint_{\mathbf{D}} \frac{\kappa_b(t)}{t - z}\, dt = -\overline{\oint_{\mathbf{B}} \frac{\kappa_b(t)}{t - \overline{z}}\, dt}.$$

Putting the pieces together, we see that for z within the racetrack Ω,

$$\kappa_b(z) = H(z) + G(z) + \overline{G(\overline{z})}. \tag{5}$$

The goal is to use equation 5 to produce a more accurate numerical approximation of $\kappa_b(z)$ from a previous approximation. We can use a Gauss-Legendre quadrature to calculate $H(z)$. For a fixed integer n, we will consider $2n$ Gauss nodes along the imaginary axis from $-i$ to i. Hence, if we have a previous approximation of $\kappa_b(z)$ for these $2n$ nodes, we can approximate $H(z)$.

Numerically approximating $G(z)$ is tricky. We can again use a Gauss-Legendre quadrature with $2n$ nodes, provided that we have a way of approximating $\kappa_b(z)$ for $\Im(z) \geq 1$. This can be done with a Fourier series. In the upper half plane, $\kappa_b(z) = \psi_b^{-1}(\rho_b^{-1}(z))$, so $\psi_b(\kappa_b(z)) - z = \rho_b^{-1}(z) - z$ will be a periodic function of period 1. By equation 2, we have that

$$\psi_b(\kappa_b(z)) - z = d_0 + d_1 e^{2\pi i z} + d_2 e^{4\pi i z} + d_3 e^{6\pi i z} + \cdots,$$

so we can compute the coefficients via

$$d_k = \int_P (\psi_b(\kappa_b(z)) - z) e^{-2k\pi i z}\, dz \tag{6}$$

where the contour P covers one period. Note that P must be in the upper half plane (the domain of $\rho_b^{-1}(z)$), within the contour Ω, and as we shall see, must be below the line $\Im(z) = 1$ to allow accurate computation of $G(z)$. Also, since $\psi_b(z)$ has a branch cut, care must be taken to ensure that the branch cut is not crossed in the integral. Thus, the optimal location for P is the line from $(-1+i)/2$ to $(1+i)/2$. The preferred integration techniques for dealing with smooth periodic functions are either the trapezoid rule or the midpoint rule [3]. We will use the midpoint method with n nodes to numerically compute the coefficients d_k. These nodes form the horizontal part of the cross in figure 2. Once we have sufficient number of coefficients, we can estimate $G(z)$ with

$$G(z) = \frac{1}{2\pi} \int_0^\pi \frac{\psi_b^{-1}(\rho_b^{-1}(i + e^{i\theta}))e^{i\theta}}{i + e^{i\theta} - z}\, d\theta,$$

using equation 2 to calculate $\rho_b^{-1}(z)$. We can now formalize the procedure for numerical calculation of $\kappa_b(z)$:

1) First, we pick a value of n. This will determine the $3n$ nodes creating the cross formation. Note that if we have one approximation $F_j(z)$ for Knesser's solution known at the $3n$ nodes, we can use the Gaussian-Legendre quadrature on equation 5 to find a better approximation $F_{k+1}(z)$ evaluated at the same $3n$ nodes. Actually, only $2n$ integrals are needed, since we can use the property that $F_{k+1}(\overline{z}) = \overline{F_{j+1}(z)}$ for the nodes below the real axis. We will label the Gauss nodes along the positive imaginary axis as y_1 to y_n. The equally spaced nodes in the horizontal part of the cross will be labeled as x_1 to x_n.

2) We will use the approximation from section 4 to create the seed values of $F_0(x_k)$ and $F_0(y_k)$. For each of the nodes y_k, we use Newton's method to find a z such that

$$\int_0^z \frac{1}{\lambda_b(t)}\, dt = y_k \tag{7}$$

using a numerical integration for each iteration step. Here, we use a high degree polynomial approximation for $\lambda_b(t)$. For y_1, we can use $z = 0$ as the initial Newton's method seed, and for other y_k we can use the

resulting z of y_{k-1}. However, equation 7 has $y_k = 0$ producing $z = 0$ instead of $z = 1$, so we will fix this by letting $F_0(y_k) \approx b^z$. In a similar way, we can compute $F_0(x_k)$ for the points on the horizontal piece of the cross. This gives us our initial approximation.

3) We now assume that we have one approximation $F_j(z)$ known at all of the $3n$ nodes of the cross. To find a better approximation, we first must find the approximate coefficients d_k assuming that $\psi_b(F_j(z)) - z$ is periodic. Since we are integrating from $(-1+i)/2$ to $(1+i)/2$, we can substitute $z = x + i/2$ to find

$$d_{k,j} = e^{k\pi} \int_{-1/2}^{1/2} (\psi_b(F_j(x + i/2)) - x - i/2) e^{-2k\pi i x}\, dx, \tag{8}$$

being careful to move the branch cut of $\psi_b(z)$ so that the branch cut is not crossed in the integral. Since we are integrating a periodic function, the midpoint rule is the best approach for numerical integration. Because this integral is only half the size of the other integrals we are doing, we only need n nodes for this integral, shown in figure 2 by the thick black equally spaced dots along the horizontal segment from $-1/2 + i/2$ to $1/2 + i/2$. Since the series in equation 3 has a radius of convergence of 1 (and in fact, it converges for most points on that circle), we know that $|d_k|$ is bounded. Because of the $e^{k\pi}$ in front of the integral, we see that the integral in equation 8 must decrease exponentially with k. Hence, we anticipate truncation errors in computing this integral for large k. However, since we are only evaluating the series in equation 2 for points with $\Im(z) \geq 1$, each coefficient d_k will be multiplied by a value of $o(e^{-2k\pi})$. Therefore, the required precision of the d_k decreases faster that the truncation errors increase.

However, there is still a limit to how many terms of the series we can compute accurately. As k increases, the number of cycles in the integrand increases, and the number of sample points is always n. We need at least three sample points per cycle if we are to avoid problems with resonance. Thus, we can only compute the first $\lfloor n/3 \rfloor$ terms of the Fourier series to any degree of accuracy. If we let

$$R_j(z) = z + \sum_{k=0}^{\lfloor n/3 \rfloor} d_{k,j} e^{2k\pi i z},$$

we find that we still should have $2n\pi/(3\ln(10)) \approx 0.9096n$ digits of precision in calculating $\kappa_b(z)$ for $\Im(z) \geq 1$.

4) We now have a way of numerically evaluating $G(z)$. Using the Gaussian quadrature with $2n$ nodes, we can approximate

$$G_j(z) = \frac{1}{2\pi} \int_0^{\pi} \frac{\psi^{-1}(R_j(i + e^{i\theta}))e^{i\theta}}{i + e^{i\theta} - z}\, d\theta.$$

Finally, we use the Gaussian quadrature again with $2n$ nodes to find the approximation of $H(z)$, which we can call $H_j(z)$. Putting the pieces together, we form

$$F_{j+1}(z) = H_j(z) + G_j(z) + \overline{G_j(\overline{z})}.$$

However, we have another complication. We need to be sure that the initial condition $F(0) = 1$ still holds. In [7], this issue was ignored until the iterations had converged, and then an adjustment was made at the end. However, we can obtain faster convergence if we incorporate the initial condition in each iteration. It is not hard to see that $\kappa_b'(0) = \lambda_b(1) = \lambda_b(0)\ln(b)$. Thus, if we let

$$\Delta a_j = \frac{F_{j+1}(0) - 1}{\lambda_b(0)\ln(b)},$$

Base	Minimum digits of precision					
b	$n=10$	$n=20$	$n=30$	$n=40$	$n=50$	$n=60$
3/2	8	16	26	33	40	50
2	8	15	24	32	41	50
e	7	14	23	33	41	49
10	6	16	24	32	41	49

Table 2: This shows the accuracy of the cross-track method for various bases and number of nodes. We used 9 key values within the contour Ω, namely $0, \pm 1/2, i/2, (i \pm 1)/2, i$, and $i \pm 1/2$, to determine the amount of accuracy within Ω.

then $F_{j+1}(-\Delta a_j)$ will be much closer to 1 than $F_{j+1}(0)$. Then we evaluate $F_{j+1}(z - \Delta a_j)$ for each of the nodes of the cross, and we are ready for the next iteration.

5) Repeat steps **3** and **4** for a fixed n until the digits stabilize to m digits of accuracy for all of the values of $F_j(x_k)$ and $F_j(y_k)$. When $b = e$, we gain a digit of accuracy for every two iterations. One could, in fact increase n dynamically (between steps 3 and 4), but for our purposes we kept n fixed so that we can compare the resulting accuracy as a function of n.

6 Numerical Results

For most values of b, the iterations quickly converge to $F_\infty(z)$, which still depends on n. Also, as n increases, the precision of $F_\infty(z)$ also increases in what seems to be a linear function of n. We explored 4 different bases of b to give a large range of behaviors. The base $b = 3/2$ is very close to the borderline case $b = e^{1/e} \approx 1.444667861$. The base $b = 2$ is special because it allows us to generalize the Ackermann function [2]. In fact, $A(4, x) = \kappa_2(x + 3) - 3$ for non-negative integers, so the tetration with base 2 allows us to define $A(4, z)$ for complex z. The base $b = e$ could be called the natural tetration, and the base $b = 10$ might be called the common tetration, since it seems to be the standard for writing extremely large numbers. If $a = \kappa_{10}^{-1}(2) \approx 0.393113$, then $\kappa_{10}(a + 1) = 100$, $\kappa_{10}(a + 2) =$ one googol, $\kappa_{10}(a + 3) =$ one googolplex, etc.

Table 2 shows how the precision of $F_\infty(z)$ varies with n for these 4 values of b. With only $n = 60$, which corresponds to 180 nodes on the cross, we obtain nearly 50 places of accuracy regardless of the base b. This is much more efficient than [7], which used over 2000 nodes to obtain 14 places of accuracy. In fact, with the exception of $b = 3/2$, the whole process can be computed in *Mathematica* in less than an hour. The issue with $b = 3/2$ is that $\psi_{3/2}(z)$ and $\psi_{3/2}^{-1}(z)$ are much harder to compute, due to the large number of iterations required to get past the "bottleneck", as described in [12]. In fact, the internal precision of the operations had to be increased to 300 digits to ensure the required accuracy of the output.

Although we have not proven that $F_\infty(z)$ does not converge to a single function as n increases, it is clear that if it does converge to a function, this function satisfies the conditions of proposition 1. Hence, this function would be Knesser's solution, $\kappa_b(z)$.

It would be prudent to compare the values of table 3 to the results of [7]. This paper computes a tetration to 14 places, and these 14 places agree with table 3 except for an occasional truncation error. For example, [7] has $F(0.1) = 1.11211143309340$, whereas rounding the result of table 3 gives $\kappa_e(0.1) = 1.11211143309341$. Otherwise, the two functions are in agreement, even for complex numbers. But this is to be expected, since by Proposition 2, the function that the method of [7] converges to, and the function the Cross-Track method converges to, are one and the same function.

z	$\kappa_{3/2}(z)$	$\kappa_2(z)$
0.0	1.000000000000000000000000000000000000000000000000000000	1.000000000000000000000000000000000000000000000000000000
0.1	1.062842348734643432422894592561194408829761235162 99	1.089118052181120252704907251320923346981511747643 38
0.2	1.121978755688757280790407791121095147771300473074 16	1.178976792567395843300163973590921672051060412867 02
0.3	1.177808364109215043535446852307098912292987424463 04	1.270145543174208663333738531909595340818242366122 15
0.4	1.230675122581517113105097002277639735729427977944 26	1.363209018045009194132395477916007936584499498959 10
0.5	1.280877277940272969013349699429730684537953093049 64	1.458781816036421700683971661038587135296606605330 91
0.6	1.328674949120326267977971893355736207848255673740 44	1.557523791625141833290155323788048174357821512353 63
0.7	1.374296225319677791772195900766104412822492149219 21	1.660157100685925367264439327112987603124236560638 27
0.8	1.417942117452334814759879126710618771246334566394 21	1.767485818836978043489927645094912838734999034949 35
0.9	1.459790609893237339440988663858398075945556476252 69	1.880419209884272735924965154213319590906272311858 96
1.0	1.500000000000000000000000000000000000000000000000000000	2.000000000000000000000000000000000000000000000000000000
z	$\kappa_e(z)$	$\kappa_{10}(z)$
0.0	1.000000000000000000000000000000000000000000000000000000	1.000000000000000000000000000000000000000000000000000000
0.1	1.112111433093407868062811548095777420924989428029 55	1.184010024624733665218767109054602787580826382 12
0.2	1.231038924931608929859971723319143199371832806242 32	1.406137583615695425772228101137295943513429719249 95
0.3	1.358383696311137608942274108361645457090352129568 35	1.680227220886396402789234849779366127692900554844 79
0.4	1.496051930399353187906599270096560282823998821175 54	2.026757028388618940551876976139523496936906565887 69
0.5	1.646354233751194580971924031592114518205311648969 04	2.477005606344964775087781705854656043202075005134 87
0.6	1.812138535701872446355777380198373539165262133421 29	3.080525271755482020497660206242617105592685434687 98
0.7	1.996971324618306847835167615097418758122999318081 9	3.919196419262728419545474246901500657237918575995 74
0.8	2.205389554552754433009853351465494365727408346248 83	5.135152840833186461087252266355267614312351140297 97
0.9	2.443257448338525254381433781945831290045634090617 26	6.989961179534714898515874991897044037670018333125 30
1.0	2.718281828459045235360287471352662497757247093699 96	10.000000000000000000000000000000000000000000000000000000

Table 3: Table of values for $\kappa_b(z)$ for various real z and various bases b.

For real values of z, the tetration can be computed using table 3. The properties $\kappa_b(z+1) = b^{\kappa_b(z)}$ and $\kappa_b(z-1) = \log_b(\kappa_b(z))$ can be used for values outside the range of the table. For complex values of z, the best way to describe the tetration is to give the Fourier coefficients for $\psi_b(\kappa_b(z)) - z$. These are listed in table 4. With these coefficients, we can compute $\kappa_b(z)$ with the formula

$$\kappa_b(z) = \psi_b^{-1}\left(z + d_0 + d_1 e^{2\pi i z} + d_2 e^{4\pi i z} + d_3 e^{6\pi i z} + d_4 e^{8\pi i z} + \cdots\right),$$

using equation 1 to compute $\psi_b^{-1}(z)$. The coefficients given can be used to compute $\kappa_b(z)$ to 25 places for $\Im(z) \geq 1$.

In [7], it was assumed that

$$\kappa_e(z) \sim L + e^{\ln(s)z+r}$$

as $\Im(z) \to \infty$ for some value of r. We can now see that $r = d_0 \ln(s)$, so for the base $b = e$,

$$\begin{aligned} r &\approx 1.07796143752792144101783319873098224011431901617802 \\ &- 0.94654096394782311971172474917056009608888185974512i \end{aligned}$$

This indicates that the results of [8] are accurate to 12 places.

Finally, since we can compute $\kappa_b(z)$ accurately for all values in the cut complex plane, we can form a contour plot of the functions. Figure 3 shows the contour plots for $b = 2$, $b = e$, and $b = 10$. The *Mathematica* notebook that generated the $\kappa_b(z)$ function, which also has the commands that generated the graphs of figure 3, can be found at

`http://myweb.astate.edu/wpaulsen/tetration.html`

Also included at this site are color versions of figure 3.

$\frac{3}{2}$	d_0:	-7.48349134602615816221240590112892949121257087361804 − 1.99996308926004674128006134398226687448913789824589i
	d_1:	0.0017571960201008128913401684803719038106943597590 8 + 0.089220694826512001436445997191820936213459969987 07i
	d_2:	0.0011497357946051570037808222681803080898776945847 5 + 0.037408388160017299292279367815753025820550025959 08i
	d_3:	0.0008553085441428287835744093170152277855019807771 6 + 0.022494174112147452325452938192435921301741498959 83i
	d_4:	0.0006826344962461548543004454440824940371833238580 5 + 0.015697829580968394187979248419667721958602268984 00i
	d_5:	0.0005689285618194070091984234275136752178506003110 6 + 0.011887071153748367185620323997333886707625320890 60i
	d_6:	0.0004882407640328371414755387586397014714979472924 + 0.009478029753449083055419644061602277372539217751 8i
	d_7:	0.0004279271306478629754113757854850812099843277 7 + 0.007830654198272824337752425328626232910815790306 i
	d_8:	0.0003810861802626298726522121553220808625937 6 + 0.006639803379088301283878603907695340450177676 1i
2	d_0:	-1.23806941407137489416696874207372877930684538882043 + 1.24816132484492036599940507052693933897696678427591i
	d_1:	0.0028646416457298654781949425090848587745979794824 8 + 0.090014162560863201546597644651413154759997047231 29i
	d_2:	0.0016361344366215896103382926595200284698885645641 2 + 0.037729617512487199013028645485412066501168125076 73i
	d_3:	0.0011518786832731889337498052484079498897488471194 6 + 0.022683070574221956862080267843278104612628451426 40i
	d_4:	0.0008905135903250987409799179255737027439814830797 5 + 0.015827492722764618005007096604591628761248447893 72i
	d_5:	0.0007264118441894542626255716510455368586347741409 0 + 0.011983969343543939244193656903123103274294844462 14i
	d_6:	0.0006136169937465890052224025336813938313920452967 + 0.009554442216383947676734462344477270453404497982 1i
	d_7:	0.0005312448144001725700417082707270735080063976776 + 0.007893188315673435338155101204325171060453030729 i
	d_8:	0.0004684141135809476704389174774561728206307295 + 0.006692386592808443356008426878462335927435927462 i
e	d_0:	-0.48841508844379660330741456880098280707763662392493 + 0.92230686292605693600282355599344017140495361014379i
	d_1:	0.0038515787354112044203993641093945858416706983535 9 + 0.090564777954116820445476674606105622980359400347 53i
	d_2:	0.0021009167869028148729866552817335141477118872724 6 + 0.037956248152475085541888936722015810780293013975 02i
	d_3:	0.0014475477584217851203220563608370288835424903050 5 + 0.022817297130023755028999729154328466016835360281 43i
	d_4:	0.0011043860992127637348984827878230583662201641185 7 + 0.015919976169422388374193454128686982804977184765 76i
	d_5:	0.0008925930993370412716873673114391469843246965762 + 0.012053223056956928649750085632871071535252072111 34i
	d_6:	0.0007487739201806153049758044951052940223070488655 0 + 0.009609109527693034937264641056222933250156351459 80i
	d_7:	0.0006447070675903776978488026882850743440031660 50 + 0.007937942295658218859174134769344880470771417123 i
	d_8:	0.0005659061641790590181344669276728723710661196 9 + 0.006730015292187770054262275187965891910776706063 i
10	d_0:	0.17799316199455913031946620533448468257225716644363 + 0.42824054091004017448062454919411317805448311007795i
	d_1:	0.0059374911244712812251020410342764605738424009846 1 + 0.091763340010080195630908672144828542040891666828926i
	d_2:	0.0030812163063365146550956049856996935941221658391 9 + 0.038461558510946910647120057622901589150232548093 57i
	d_3:	0.0020685320149818848591154758942364769056055270270 3 + 0.023122093602265423730534776575281605503259780128 31i
	d_4:	0.0015519657819802903502588809303251894919390850395 3 + 0.016133167185261008358090702937214364723817454920 27i
	d_5:	0.0012393355977077787601160843261674070280919906186 0 + 0.012214933338069346004942430688176700647431882771 36i
	d_6:	0.0010300792001943083139326767604951363050147478562 2 + 0.009738210267180334276342023872893324511605863133 82i
	d_7:	0.0008803531087551340517600862153381756524490261 00 + 0.008044705139249328265038879933826000552442989948 1i
	d_8:	0.0007680061010486093859783820143013718422587605 4 + 0.006820606592296152855983866087877088167089753177 i

Table 4: This table shows the coefficients d_i's for various bases. The base b is indicated on the left. In all cases, the coefficients slowly converge to 0, since the series in equation 3 has a radius of convergence of 1.

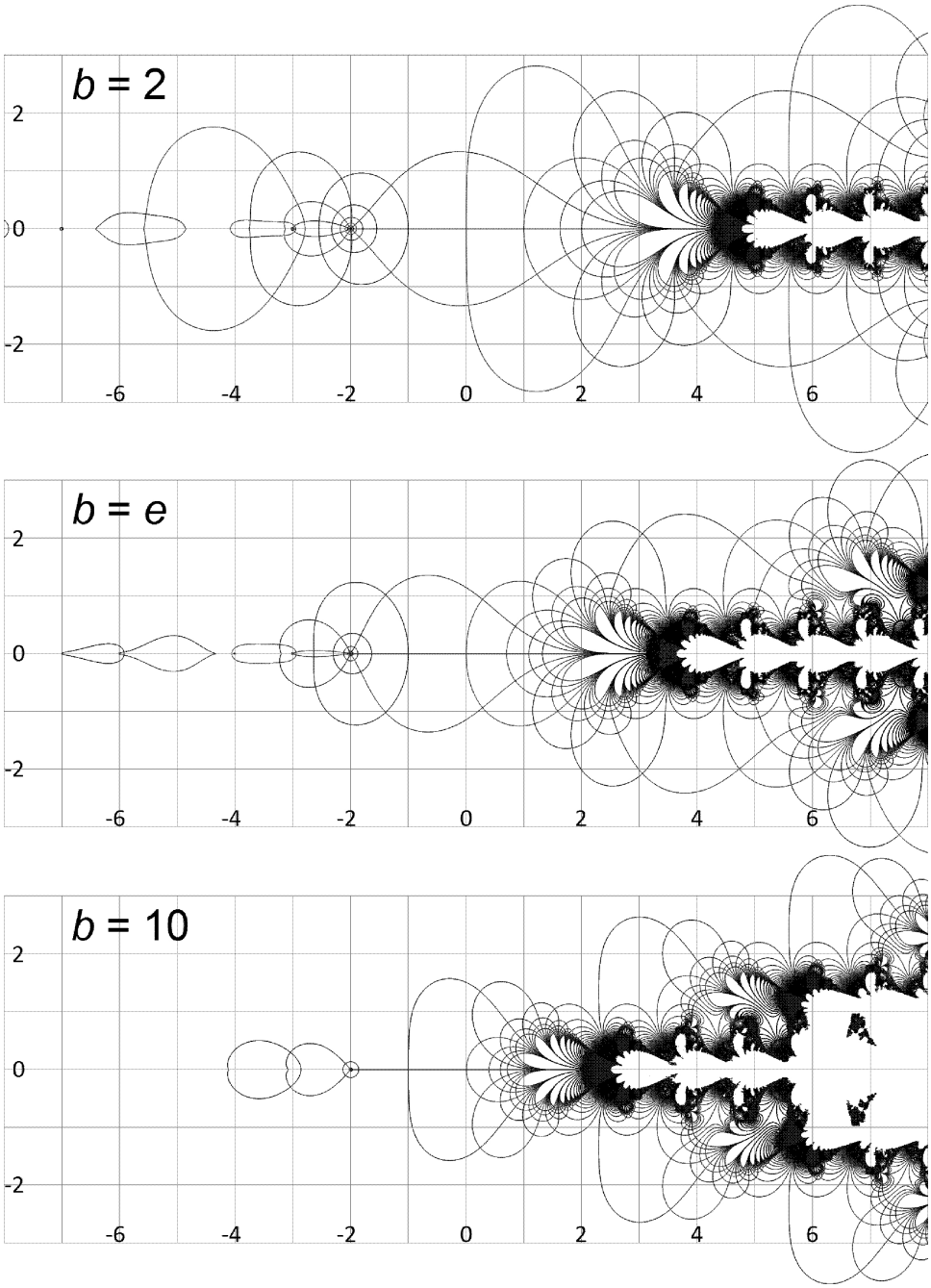

Figure 3: Level curves for $\Re(\kappa_b(z))$ and $\Im(\kappa_b(z)) = 0, \pm 1, \pm 2, \pm 3, \pm 4$ are shown for various b.

7 Conclusion

We have found a way to numerically approximate $\kappa_b(z)$ to arbitrarily high degree of accuracy. One could produce 100 digits of precision in a few hours. But the technique introduced here can be used to solve many other equations. Most functional equations can be reduced to solving Abel's equation $\alpha(g(z)) = \alpha(z) + 1$ for a given function $g(z)$ [11]. The cross-track method could be used to calculate $\alpha^{-1}(z)$, which in turn can be used to calculate $\alpha(z)$.

But this technique also opens the door for new discoveries in the field of tetration. We saw in Corollary 2 that for a fixed real x_0, $\kappa_b(x_0)$ can be thought of as a real analytic function of b. Then there is a unique way of extending this function into the complex plane. Hence, we could define $\kappa_b(z)$ for *complex* b, (although there are bound to be logarithmic singularities appearing). This would give us a firm definition of complex tetration.

There is also no reason why we cannot use the cross-track method to solve $F(z+1) = \kappa_b(F(z))$. This would give us a complex *pentation*, which is repeated tetration. We would need to first compute the power series for $\kappa_b(z)$ at a fixed point, and it would be harder to come up with an initial approximation. This was actually done in [9] for base e, but perhaps this can be explored in a future paper with different bases.

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