1.5 Nested Quantifiers (N. Qs)

- N. Qs are nested if one Q is within the scope of another.

**Example:** \( \forall x \exists y (x + y = 1) \) can be easily understood as

\[
\forall x \ Q(x) \\
Q(x) \ is \ \exists y \ P(x, y) \\
P(x, y) \ is \ (x + y = 1)
\]

**Example 1**

Translate into English the following statements. \( D = \mathbb{R} \).

1. \( \forall x \forall y (x + y = y + x) \). (2) \( \forall x \forall y ((x > 0) \land (y > 0) \rightarrow (xy > 0)) \).

**Example 2**

Determine the truth value of the following nested quantifications, \( D = \mathbb{Z} \).

1. \( \forall n \exists m \ (n^2 \leq m) \) (2) \( \exists n \forall m \ (nm = m) \) (3) \( \exists n \exists m \ (n^2 + m^2 = 5) \) (4) \( \forall n \forall m \exists p \ (p = (m + n)/3) \)
The Order of Universal (Existential) Quantifiers

Example 3

1. Let $P(x, y) : x + y = y + x$. Tell the truth values of $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$, where $D = \mathbb{R}$.

2. Let $P(x, y) : x + y = 2$. Tell the truth values of $\exists x \exists y P(x, y)$ and $\exists y \exists x P(x, y)$, where $D = \mathbb{Z}$.

- From the previous example, we can see that even if the order of nested universal (existential) quantifiers in a statement is changed, the meaning of the quantified statements will be the same.
The Order of mixtures of U. and E. Quantifiers

- The order of nested existential and universal quantifiers in a statement has to be considered with more careful treatment.

**Example 4**

1. Let $Q(x, y) : x + y = 1$. Tell the truth values of $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where $D = \mathbb{R}$.
2. Let $P(x, y, z) : x + y = z$. Tell the truth values of $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y P(x, y, z)$, where $D = \mathbb{R}$.

- As we did in the previous example, switching the order of nested existential and universal quantifiers in a statement may make a difference. But switching the order may not make a difference. Here is an example.

**Example 5**

Let $Q(x, y) : xy = 1$. Tell the truth values of $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where $D = \mathbb{R}$.
The following table will summarize the meanings of different possible quantifications.

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True (T)?</th>
<th>When False (F)?</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀x∀yP(x, y)</td>
<td>P(x, y): T for every pair x, y</td>
<td>There is a pair x, y for which P(x, y): F</td>
</tr>
<tr>
<td>∀y∀xP(x, y)</td>
<td>For every x there is y for which P(x, y): T</td>
<td>There is an x such that P(x, y): F for every y</td>
</tr>
<tr>
<td>∀x∃yP(x, y)</td>
<td>There is an x for which P(x, y): T for every y</td>
<td>For every x there is a y for which P(x, y): F</td>
</tr>
<tr>
<td>∃x∀yP(x, y)</td>
<td>There is a pair x, y for which P(x, y): T</td>
<td>P(x, y): F for every pair x, y</td>
</tr>
<tr>
<td>∃y∃xP(x, y)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From Nested Quantifiers into English

Example 6

1. Let \( D = \{(x, y) \mid x, y \text{ are students in ASU}\} \) be the domain. Then, translate into English the statement
\[
\forall x (C(x) \lor \exists y (C(y) \land F(x, y))),
\]
where \( C(x) \) : “\( x \) has a computer”, \( F(x, y) \) is “\( x \) and \( y \) are friends”.

2. \( Q(x, y) : x \) has sent an email message to \( y \), where
\( D = \{(x, y) \mid x, y \text{ are students in ASU}\} \). Express the followings in English. (1) \( \exists x \exists y Q(x, y) \) (2) \( \forall y \exists x Q(x, y) \)

3. Translate the following nested quantifications into English, where \( D = \mathbb{R} \).
(1) \( \exists x \forall y (xy = y) \) (2) \( \forall x \forall y ((x < 0) \land (y < 0)) \rightarrow (xy > 0) \)
From English into Nested Quantifiers

Example 7

Express the following statements as a logical expression involving predicates, quantifiers, and logical connectives.

(1) Everyone has exactly one best friend. \(D\) is a set of all people.
(2) All math and computer science students need to take Discrete Structures. \(D\) is a set of C.S. and math students.
(3) Every student in our class has taken at least one mathematics course.

\[ D = \{(x, y) | x \text{ is a student in our class and } y \text{ is a math course}\}. \]