2.4 Sequences and Summations

- **Sequences vs Sets**
  Sequences are **ordered** lists of elements, while sets are collections of elements regardless of order.

- A finite sequence: \( \{a_1, a_2, \cdots, a_n\} \) (called a string) and an infinite sequence: \( \{a_1, a_2, \cdots, a_n, \cdots\} \).

**Definition**

A sequence is a **function** \( f \) from \( A \subset \mathbb{Z} \) to a set \( S \). So the image of the integer \( n \) is denoted by \( a_n = f(n) \). \( a_n \) is called a **general term** of the sequence. The sequence is denoted by \( \{a_n\} \).

**Example 1**

Write the first five terms of the sequence \( \{a_n\} \) defined by \( a_n = (-1)^n/n \). for \( n \geq 1 \).
Definitions

1. A geometric progression is a sequence of the form $a_n = ar^{n-1}$ for $n \geq 1$, where $a$ is called the initial term and $r$ is the common ratio. So $r = a_{n+1}/a_n$.

2. An arithmetic progression is a sequence of the form $a_n = a + (n - 1)d$ for $n \geq 1$, where $a$ is called the initial term and $d$ is the common difference. So $d = a_{n+1} - a_n$.

Example 2

1. Let $a_n = 2 \cdot (-3)^n + 5^n$. Then find (1) $a_0$ (2) $a_4$.

2. Let $a_n = 2n + 1$. Then find (1) $a_1$ (2) $a_3$.

3. Let $a_n = (n + 1)^{n+1}$. Then find (1) $a_0$ (2) $a_2$.

4. Let $a_n = \lfloor n/2 \rfloor$. Then find (1) $a_1$ (2) $a_3$. 
**Definition**

A **recurrence relation** for \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more of the previous terms for \( n \geq n_0 \geq 0 \).

**Example 3**

For \( n \geq 1 \)

1. let \( a_n = a_{n-1} + 3 \). If \( a_0 = 2 \), then find (1) \( a_1 \) (2) \( a_3 \).
2. let \( a_n = a_{n-1}^2 \). If \( a_0 = 2 \), then find the next five terms.
3. let \( a_{n+1} = a_n + a_{n-1} \). (Fibonacci sequence) If \( a_0 = 0 \) and \( a_1 = 1 \), then find the next five terms.
We use the $\sum$ notation to express the sum of finitely (infinitely) terms:

\[
a_m + a_{m+1} + \cdots + a_n = \sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{k=m}^{n} a_k = \cdots,
\]

where $j$ is the index of summation and $m$ is the lower limit and $n$ is the upper limit.

**Example 4**

What are the values of the following sums?

1. $\sum_{k=1}^{5} (k + 1)$
2. $\sum_{i=1}^{10} 3$
3. $\sum_{j=0}^{8} (2^{j+1} - 2^j)$.

**Example 5**

Compute the following double sums

1. $\sum_{i=1}^{2} \sum_{j=1}^{3} (2i + 3j)$
2. $\sum_{i=0}^{2} \sum_{j=1}^{3} ij$
Some Useful Summation Formula

<table>
<thead>
<tr>
<th>Sum</th>
<th>Closed form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=0}^{n} ar^k \ (r \neq 0)$</td>
<td>If $n \neq \infty$, $a(r^{n+1} - 1)/(r - 1)$ for $r \neq 1$,</td>
</tr>
<tr>
<td></td>
<td>If $n = \infty$, $-a/(r-1)$ for $</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^2$</td>
<td>$\frac{n(n+1)(2n+1)}{6}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^3$</td>
<td>$\left(\frac{n(n+1)}{2}\right)^2$</td>
</tr>
</tbody>
</table>

Example 5

Find the following sums
1. $\sum_{k=5}^{10} k$ (2) $\sum_{i=3}^{5} k^3$

Example 6

1. Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$.
   This type of sum is called telescoping sum.
2. Find $\sum_{j=1}^{10} \frac{1}{j(j+1)}$. We need to use telescoping sum.