3.2 Growth of Functions

- When we analyze some algorithms, we may want to consider the number of steps. For example, assume that the number of steps to complete a mathematical problem of size \( n \) is given by \( f(n) = 3n^2 + 4n - 1 \). If we ignore constants and slower growing terms, we can say that \( f(n) \) depends on the dominating term \( n^2 \) for sufficiently large numbers \( n \). How can we describe it, using a mathematical notation? The growth of functions can be written using big-\( O \) notation.

**Definition**

Let \( f \) and \( g \) be functions with the domain \( \mathbb{R} \). We say that \( f(x) = O(g(x)) \) if there is a constant \( C > 0 \) and \( k \in \mathbb{R} \) such that \( |f(x)| \leq C |g(x)| \), provided that \( x \geq k \). \( C \) and \( k \) are called witnesses.
The growth of functions commonly used in Big-O estimates. Have a look at Figure 3 in page 211.

**Theorem 1**

Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), where \( a_i \in \mathbb{R} \) for \( 0 \leq i \leq n \). Then \( P(x) = O(x^n) \).

There are three important rules to find big-O notation.

**Theorem 2**

1. Suppose that \( f_1(x) = O(g_1(x)) \) and \( f_2(x) = O(g_2(x)) \). Then \( (f_1(x) + f_2(x)) = O(\max(|g_1(x)|, |g_2(x)|)) \).
2. Suppose that \( f_1(x) = O(g(x)) \) and \( f_2(x) = O(g(x)) \). Then \( (f_1(x) + f_2(x)) = O(g(x)) \).
3. Suppose that \( f_1(x) = O(g_1(x)) \) and \( f_2(x) = O(g_2(x)) \). Then \( (f_1(x)f_2(x)) = O(g_1(x)g_2(x)) \).
Examples

In problems 1-4, find witnesses $C$ and $k$.

1. Let $f(x) = x^2 + 2x + 1$. Show that $f(x) = O(x^2)$.
2. Determine whether the followings can make $f(x) = O(x)$.
   (1) $f(x) = 10^{10}x + 1$  (2) $f(x) = 10^{-6}x^2 + x + 2$  (3) $f(x) = 10\log x$
   (4) $f(x) = \lfloor x \rfloor$  (5) $f(x) = \lceil x/2 \rceil$
3. Let $f(x) = (x^2 + 1)/(x + 1)$. Then show that $f(x) = O(x)$.
4. Find the least integer $n$ such that $f(x) = O(x^n)$.
   (1) $f(x) = 2x^3 + x^2\log x$  (2) $f(x) = 3x^3 + (\log x)^4$
   (3) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
   (4) $f(x) = (x^4 + 5\log x)/(x^4 + 1)$
5. Find the best corresponding big-$O$ notation.
   (1) $(n^2 + 8)(n + 1)$  (2) $(n\log n + n^2)(n^3 + 2)$
   (3) $(n! + 2^n)(n^3 + \log(n^2 + 1))$  (4) $n\log(n^2 + 1) + n^2\log n$
   (5) $(n\log(n + 1))^2 + (\log(n + 1))(n^2 + 1)$  (6) $n^{2^n} + n^{n^2}$.