9. Relations

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In our everyday life, we may consider relationships between an employees and their salary, people and their relatives, and so on. In mathematics, we study more clear relationships between numbers, and between a input $x$ and a value of a function $f(x)$, and so on.

Relationships between elements of sets are represented, using a subset of the Cartesian product of the sets.

An equivalence relation which is a special type of relations arises throughout mathematics and computer science.
The mathematical way to express a relationship between elements of two sets is to use ordered pairs.

**Definition**

A **binary relation** from $A$ to $B$ is a subset of $A \times B$.

- From the definition, a binary relation from $A$ to $B$ is $R = \{(a, b) \mid a \in A \land b \in B\} \subseteq A \times B$.
- The notation $a R b$ means that $(a, b) \in R$ and $a (\sim R) b$ means that $(a, b) \notin R$.
- When $(a, b) \in R$, $a$ is said to be related to $b$ by $R$.

**Example 1**

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then a possible relation from $A$ to $B$ is $\{(0, a), (0, b), (1, b), (2, a)\}$. Then $0 R b$ and $1 (\sim R) a$.

- Relations can be represented graphically. We use arrows to represent ordered pairs and use a table to represent relations.
Example 2

List the ordered pairs in the relation \( R \) from \( A = \{0,1,2,3,4\} \) to \( B = \{0,1,2\} \), where \((a, b) \in R\) if and only if

1. \( a = b \)
2. \( a \mid b \)
3. \( \text{lcm}(a, b) = 2 \).

- A function \( f \) from \( A \) to \( B \) can be understood, based on a relation from \( A \) to \( B \), since the graph of \( f \) is \( \{(a, b) \mid b = f(a)\} \). So relations are a generalization of graphs of functions.

Definition

A relation on a set \( A \) is a relation from \( A \) to \( A \).

- In another word, a relation \( R \) on a set \( A \) is a subset of \( R \subseteq A \times A \).

Example 3

Let \( A = \{0,1,2\} \). Which ordered pairs are in the relation \( R = \{(a, b) \mid a \mid b\}? \)
Properties of Relations: let $A$ be a set.

**Definitions**

1. A relation $R$ on $A$ is called **reflexive** if $(a, a) \in R$ for $\forall a \in A$.
2. A relation $R$ on $A$ is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for $\forall a, b \in A$.
3. A relation $R$ on $A$ is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$ for $\forall a, b \in A$.
4. A relation $R$ on $A$ is called **transitive** if $(a, c) \in R$ whenever $(a, b) \in R$ and $(b, c) \in R$ for $\forall a, b, c \in A$.

**Example 4**

For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

(1) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
(2) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
(3) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
Example 5

Determine whether the relation $R$ on $\mathbb{Z}$ is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

1. $x \neq y$
2. $xy \geq 1$
3. $x \equiv y \pmod{7}$
4. $x \geq y^2$

Combining Relations

Example 6

Let $R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$, $R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}$, $R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, $R_4 = \{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$, $R_5 = \{(a, b) \in \mathbb{R}^2 \mid a = b\}$, and $R_6 = \{(a, b) \in \mathbb{R}^2 \mid a \neq b\}$.

Then find (1) $R_1 \cup R_2$ (2) $R_4 \cap R_6$ (3) $R_6 - R_3$